# Nonlinear elasticity of the sliding columnar phase

C. S. O'Hern* and T. C. Lubensky
*Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, Pennsylvania 19104*
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The sliding columnar phase is a liquid-crystalline phase of matter composed of two-dimensional smectic lattices stacked one on top of the other. This phase is characterized by strong orientational but weak positional correlations between lattices in neighboring layers and a vanishing shear modulus for sliding lattices relative to each other. A simplified elasticity theory of the phase only allows intralayer fluctuations of the columns and has three important elastic constants: the compression, rotation, and bending moduli $B$, $K_y$, and $K$. The rotationally invariant theory contains anharmonic terms that lead to long-wavelength renormalizations of the elastic constants similar to the Grinstein-Pelcovits renormalization of the elastic constants in smectic liquid crystals [Phys. Rev. Lett. **47**, 856 (1981); Phys. Rev. A **26**, 915 (1982)]. We calculate these renormalizations at the critical dimension $d=3$ and find that $K_y(q)\sim K^{1/2}(q)\sim B^{-1/3}(q)\sim[\ln(1/q)]^{1/4}$, where $q$ is a wave number. The behavior of $B$, $K_y$, and $K$ in a model that includes fluctuations perpendicular to the layers is identical to that of the simple model with rigid layers. We use dimensional regularization rather than a hard-cutoff renormalization scheme because ambiguities arise in the one-loop integrals with a finite cutoff. [S1063-651X(98)11011-5]

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## I. INTRODUCTION

DNA, which is a semiflexible polymer, and cationic lipids in solution form complexes in which the negative charge of the DNA is nearly compensated for by the positive charge of the lipids. These complexes are under intensive study as possible nonviral carriers of DNA to cell nuclei for gene therapy [1]. Rädler *et al.* have shown that under appropriate conditions the complexes self-assemble into multilamellar structures [2]. The lipids form stacked bilayer sheets with DNA molecules intercalated in the galleries between the bilayers as shown in Fig. 1. Each gallery is thick enough to accommodate only one DNA molecule and its hydration layer. Within each gallery, DNA molecules adopt a linear rather than a coiled configuration and form a regularly spaced parallel array that in the absence of couplings to DNA in other galleries is a two-dimensional smectic liquid crystal [3]. The experimentally determined x-ray structure factor of these complexes is well modeled by a stack of weakly coupled two-dimensional (2D) smectic lattices [2].

Two recent theoretical papers [4,5] have pointed out that weakly coupled 2D smectic lattices form a different phase of matter, the *sliding columnar* phase. This phase is characterized by strong orientational correlations but weak positional correlations between smectic lattices. All lattices are aligned on average along a common direction (the $x$ axis in Fig. 1), but their relative positions decorrelate exponentially with distance between smectic lattices. With sufficiently strong coupling between galleries, long-range positional correlations between smectic layers develop and the system becomes an anisotropic columnar phase with a two-dimensional DNA lattice in the plane perpendicular to the direction of DNA alignment. The sliding columnar phase, on the other hand, is what the columnar phase becomes when coupling between galleries becomes so weak that DNA lattices can slide freely across each other. It has no shear modulus resisting relative displacements of DNA lattices, but it does have a rotation modulus resisting their relative rotation. Dislocations may melt the sliding columnar phase to an anisotropic nematic lamellar phase at length scales longer than an in-plane dislocation unbinding length [6]. It is possible, however, to choose interlayer interactions so that the sliding columnar phase is the stable equilibrium phase at all length scales [7].

This paper will investigate the nonlinear elasticity of the sliding columnar (SC) phase. Its principal purpose is to show that the nonlinear strains lead to a Grinstein-Pelcovits renormalization of the elastic constants [8] and not, as one could imagine, to the destruction of the sliding columnar phase itself. The lipid bilayers, which we take to be aligned on average parallel to the $xz$ plane as shown in Fig. 1, fluctuate like bilayers in any lamellar phase. To understand correlations and fluctuations of the DNA smectic lattices, it is convenient to consider first a model in which the lipid bilayers are rigid planes with no fluctuations in the $y$ direction. In this

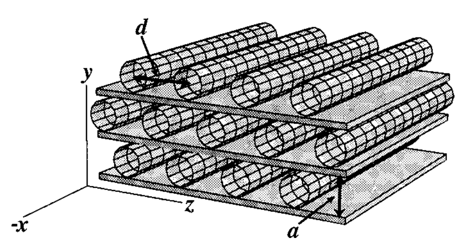


FIG. 1. Picture of the idealized sliding columnar phase. The DNA columns are sandwiched between planar lipid bilayer sheets. The bilayer planes are stacked in the $y$ direction with spacing $a$. The DNA columns are oriented in the $x$ direction and within each layer the columns are separated by $d$. The positions of columns in neighboring layers are uncorrelated.

*Author to whom correspondence should be addressed. Electronic address: ohern@lubensky.physics.upenn.edu

 

TABLE I. Comparison of the logarithmic scaling exponents for the elastic moduli of the 3D smectic and sliding columnar phases. At long wavelengths the elastic moduli for both phases scale as $\ln^{\alpha}[1/q]$, with $\alpha$ given below.

| Phase | $B$ | $K$ | $K_y$ |
|---|---|---|---|
| 3D smectic | −4/5 | 2/5 | |
| Sliding columnar | −3/4 | 1/2 | 1/4 |

case, displacements of the DNA lattices, which are aligned on average along the $x$ direction, are restricted to the $z$ direction.

The rotationally invariant Landau-Ginzburg-Wilson Hamiltonian in units of $k_BT$ for this system is

$$\mathcal{H}=\frac{1}{2}\int d^3x[Bu_{zz}^2+K_y(\partial_x\partial_y u_z)^2+K(\partial_x^2 u_z)^2], \tag{1}$$

where $B$, $K_y$, and $K$ are the compression, rotation, and bending moduli divided by $k_BT$ and

$$u_{zz}=\partial_z u_z-\frac{1}{2}[(\partial_x u_z)^2+(\partial_z u_z)^2] \tag{2}$$

is the nonlinear Eulerian strain appropriate for the sliding columnar phase. Note that $\mathcal{H}$ is invariant under

$$u_z'(\mathbf{x})\rightarrow u_z(\mathbf{x})+f(y). \tag{3}$$

It is this fact that ensures that nonlinearities do not destroy the sliding columnar phase.

The rotationally invariant strain $u_{zz}$ introduces anharmonic terms into the Hamiltonian that lead to a Grinstein-Pelcovits renormalization of $B$, $K_y$, and $K$. The renormalized moduli scale logarithmically with $q$ at long wavelengths:

$$K_y(q)\sim K^{1/2}(q)\sim B^{-1/3}(q)\sim\left[\ln\left(\frac{\mu}{q}\right)\right]^{1/4}, \tag{4}$$

where $\mu$ is a large momentum cutoff. A complete model for the sliding columnar phase allows both lipid bilayers and smectic lattices to fluctuate. This model also exhibits Grinstein-Pelcovits renormalization of the elastic constants. Table I lists the exponents describing singularities in the elastic constants for both the 3D smectic and sliding columnar phases.

The evaluation of the above renormalization presented some unexpected difficulties. The continuum Hamiltonian in Eq. (1) is formally invariant under arbitrary global rotations. However, the introduction of a hard cutoff breaks this rotational invariance just as the introduction of a similar cutoff breaks gauge invariance in gauge Hamiltonians [9]. Nevertheless, hard-cutoff renormalization group (RG) procedures can with care be applied successfully to Hamiltonians with gauge [10] or rotation symmetries [11]. Indeed, the original Grinstein-Pelcovits calculation of the logarithmic renormalization of the smectic-$A$ elastic constants used a hard cutoff [8]. When we applied the popular momentum-shell hard-cutoff RG procedure [12] to the nonlinearities in the sliding columnar phase, we encountered ambiguities that we were unable to resolve. We found that the values of the one-loop diagrams depended on whether the external momentum was added to the top or the bottom part of the internal loop. Similar difficulties are not encountered in the Grinstein-Pelcovits calculation. To eliminate these ambiguities, we switched to the dimensional regularization procedure that explicitly preserves rotational invariance because the cutoffs are infinite [13].

The remainder of the paper will be organized as follows. We first rederive the results of Grinstein and Pelcovits in Sec. II using dimensional regularization. Then, in Sec. III we calculate the renormalization of the sliding columnar elastic constants of the simplified theory using the same scheme. In Sec. IV we relax the constraint of rigid membranes and show that the membrane fluctuations do not modify the scaling behavior of the elastic moduli of the rigid theory. We give a brief conclusion in Sec. V. In Appendixes A and B, we evaluate the one-loop diagrams for the 3D smectic and simplified sliding columnar theories. In Appendix C we show that ambiguities arise when a finite cutoff is implemented to calculate the loop diagrams of the sliding columnar theory. Finally, in Appendix D we derive the nonlinear strains required for the rotationally invariant theory of the sliding columnar phase in the presence of fluctuating membranes.

## II. RG ANALYSIS OF THE 3D SMECTIC LIQUID CRYSTAL

The rotationally invariant elasticity theory for a smectic liquid crystal contains nonlinear terms that renormalize the elastic constants of the harmonic theory for all dimensions below 3. Grinstein and Pelcovits calculated the corrections to the elastic constants of a 3D smectic using a RG analysis with a finite-wave-number cutoff [8]. They found that the corrections to both the compression and bending moduli are logarithmic in the wave number $q$ with the former scaling to zero and the latter scaling to infinity at long wavelengths. Application of a hard-cutoff RG procedure to the sliding columnar phase leads to ambiguities with no obvious resolution. (See Appendix C.) We therefore employ a dimensional regularization procedure that sends the cutoff to infinity and thereby preserves rotational invariance. In this section we rederive the Grinstein-Pelcovits results for a 3D smectic using dimensional regularization. This establishes the language needed to calculate the renormalization in the sliding columnar phase.

### A. Rotationally invariant theory

A smectic liquid crystal in $d$ dimensions is characterized by a mass-density wave with period $P=2\pi/q_0$ along one dimension and by fluidlike order in the other $d-1$ dimensions. The phase of the mass-density wave at the point $\mathbf{x}=(\mathbf{x}_\perp,z)$ is $q_0[z-u(\mathbf{x})]$. The elastic Landau-Ginzburg-Wilson Hamiltonian for a smectic is identical in form to the Landau–Peierls–deGennes Hamiltonian for a 1D solid. In units of $k_BT$, it is

$$\mathcal{H}=\frac{1}{2}\int d^dx[B_{\rm sm}u_{zz}^2+K_{\rm sm}(\nabla_\perp^2 u)^2], \tag{5}$$

where $\nabla_\perp$ is the gradient operator in the $d-1$ subspace spanned by $\mathbf{x}_\perp$, and $B_{\rm sm}$ and $K_{\rm sm}$ are, respectively, the compression and bending moduli divided by $k_BT$. The nonlinear Eulerian strain $u_{zz}=\partial_z u-(1/2)(\nabla u)^2$ is invariant with respect to uniform, rigid rotations of the smectic layers. Below we will drop the $(\partial_z u)^2$ term in $u_{zz}$ since its inclusion leads to nonlinear terms that are irrelevant in the RG sense with respect to the two quadratic terms in Eq. (5). Therefore, we will take

$$u_{zz}\approx\partial_z u-\frac{1}{2}(\nabla_\perp u)^2. \tag{6}$$

Strictly speaking, we should include a term linear in $u_{zz}$ whose coefficient is chosen to make $\langle u_{zz}\rangle=0$. The inclusion of and proper treatment of this term do not modify our RG equations and we will ignore it here and in our treatment of the sliding columnar phase.

### B. Engineering dimensions

To implement our RG procedure it is convenient to rescale parameters so that $B_{\rm sm}$ is replaced by unity and the nonlinear form of $u_{zz}$ is preserved. To this end, we scale $u$ and $\mathbf{x}$ as

$$u=L_u\tilde{u},\quad z=L_z\tilde{z},\quad \mathbf{x}_\perp=\tilde{\mathbf{x}}_\perp. \tag{7}$$

Note that $\mathbf{x}_\perp$ does not rescale. Under these rescalings we obtain

$$u_{zz}=L_uL_z^{-1}\left(\partial_{\tilde{z}}\tilde{u}-\frac{1}{2}L_uL_z(\nabla_{\tilde{\perp}}\tilde{u})^2\right). \tag{8}$$

We require $u_{zz}=A\tilde{u}_{zz}$, where $\tilde{u}_{zz}=\partial_{\tilde{z}}\tilde{u}-(1/2)(\nabla_{\tilde{\perp}}\tilde{u})^2$ is the rescaled nonlinear strain with the same form as Eq. (6). This yields $L_u=L_z^{-1}$ and $A=L_u^2$. The coefficient of $\tilde{u}_{zz}^2$ in the rescaled Hamiltonian is set to one with the choice

$$L_u=B_{\rm sm}^{-1/3}. \tag{9}$$

The rescaled theory then becomes

$$\mathcal{H}=\frac{1}{2}\int d^d\tilde{x}\left[\tilde{u}_{zz}^2+\frac{1}{w}(\nabla_{\tilde{\perp}}^2\tilde{u})^2\right], \tag{10}$$

with

$$w=\frac{B_{\rm sm}^{1/3}}{K_{\rm sm}}. \tag{11}$$

For the remainder of Sec. II we will use the Hamiltonian in Eq. (10) but drop the tilde on the scaled variables.

We determine the dimensions of the scaled variables using the engineering dimensions of $B_{\rm sm}$ and $K_{\rm sm}$. The dimension $d_A$ determines how $A$ scales with length $L$: $[A]=L^{d_A}$. From the respective dimensions $d_{B_{\rm sm}}=-d$ and $d_{K_{\rm sm}}=2-d$ of $B_{\rm sm}$ and $K_{\rm sm}$, we obtain $[L_u]=[L_z^{-1}]=L^{d/3}$. Using these we find the following for the dimensions of the scaled variables and the parameter $w$:

$$[u]=\left[\frac{L}{L_u}\right]=L^{\epsilon/3},\quad [z]=\left[\frac{L}{L_z}\right]=L^{1+d/3},$$
$$[x_\perp]=L,\quad [w]=\left[\frac{L^{-d/3}}{L^{2-d}}\right]=L^{-2\epsilon/3}, \tag{12}$$

where $\epsilon=3-d$. Using these definitions one can easily verify that both terms in Eq. (10) are dimensionless. $[w]$ scales as $\mu^{2\epsilon/3}$ where $[\mu]=L^{-1}$ and it is therefore a relevant variable below $d=3$. The dimensions of the coefficients of the $(\partial_z u)^3$, $(\partial_z u)^2(\nabla_\perp u)^2$, and $(\partial_z u)^4$ terms are $2d/3$, $2d/3$, and $4d/3$, respectively. These nonlinear terms are irrelevant and will be ignored in what follows.

The engineering dimensions in Eq. (12) imply that there is an invariance of $\mathcal{H}$ under the transformation $\mu\to\mu b$ and

$$u(\mathbf{x}_\perp,z)=b^{d_u}u'(\mathbf{x}'_\perp,z'), \tag{13}$$

where $\mathbf{x}'_\perp=b^{-1}\mathbf{x}_\perp$ and $z'=b^{-(1+d/3)}z$, i.e.,

$$\mathcal{H}[u,w,\mu]=\mathcal{H}[u',wb^{2\epsilon/3},\mu b]. \tag{14}$$

This in turn implies a scaling form for the position correlation function $G(\mathbf{x}_\perp,z)=\langle u(\mathbf{x}_\perp,z)u(0,0)\rangle$ and its Fourier transform $G(\mathbf{q})$. We find

$$G(\mathbf{x}_\perp,z,w)=b^{2(1-d/3)}G(\mathbf{x}'_\perp,z',wb^{2\epsilon/3}) \tag{15}$$

and from this we obtain the vertex function $\Gamma(\mathbf{q})=G^{-1}(\mathbf{q})$,

$$\Gamma(\mathbf{q}_\perp,q_z,w)=b^{-2(1+d/3)}\Gamma(b\mathbf{q}_\perp,b^{1+d/3}q_z,wb^{2\epsilon/3}). \tag{16}$$

When $d=3$ this reduces to the scaling form

$$\Gamma(\mathbf{q}_\perp,q_z,w)=q_\perp^4\Gamma\left(1,\frac{q_z}{q_\perp^2},w\right), \tag{17}$$

which the harmonic vertex function $\Gamma=q_z^2+w^{-1}q_\perp^4$ satisfies.

### C. RG procedure

To calculate renormalized quantities, we seek a multiplicative procedure that yields a renormalized Hamiltonian with the same form as the original Hamiltonian, i.e., a Hamiltonian that is a function of a renormalized nonlinear strain with the same form as Eq. (6). To preserve the form of the strains, it is necessary to rescale fields and lengths simultaneously. The rescaling that produced Eq. (10) shows that the form of $u_{zz}$ is preserved if the rescaling coefficients of $u$ and $z$ are inverses of each other. We therefore introduce a renormalization constant $\mathcal{Z}$ and a renormalized displacement $u'$ such that

$$u(\mathbf{x})=\mathcal{Z}^{1/3}u'(\mathbf{x}')=\mathcal{Z}^{1/3}u'(\mathbf{x}_\perp,\mathcal{Z}^{1/3}z). \tag{18}$$

This implies that $u_{zz}(\mathbf{x})=\mathcal{Z}^{2/3}u'_{zz}(\mathbf{x}')$. We also introduce a unitless renormalized coupling constant $g$ and renormalization constant $\mathcal{Z}_g$ via

$$w^{3/2}=g\mu^\epsilon\mathcal{Z}_g\mathcal{Z}^{1/2}, \tag{19}$$

where $\mu$ is an arbitrary wave-number scale. The renormalized Hamiltonian then becomes

$$\mathcal{H}' = \frac{1}{2}\int d^dx'[\mathcal{Z}(u'_{zz})^2 + (g\mu^{\epsilon}\mathcal{Z}_g)^{-2/3}(\nabla_{\perp}^2,u')^2]. \tag{20}$$

We now follow standard procedures to evaluate $\mathcal{Z}(g)$ and $\mathcal{Z}_g(g)$ [13]. The renormalized Hamiltonian in Eq. (20) determines the vertex function

$$\Gamma(\mathbf{q}) = q_z^2 + (g\mu^{\epsilon})^{-2/3}q_{\perp}^4 + (\mathcal{Z}-1)q_z^2 + (g\mu^{\epsilon})^{-2/3}(\mathcal{Z}_g{}^{-2/3}-1)q_{\perp}^4 + \Sigma(\mathbf{q}) \tag{21}$$

to one-loop order, where $\Sigma(\mathbf{q})$ is the one-loop diagrammatic contribution to $\Gamma(\mathbf{q})$. We next impose the following conditions on the vertex function to enforce the correct scaling behavior:

$$\left.\frac{d\Gamma}{dq_z^2}\right|_{q_z=\mu^2,q_{\perp}=0} = 1, \tag{22a}$$

$$\left.\frac{d\Gamma}{dq_{\perp}^4}\right|_{q_z=\mu^2,q_{\perp}=0} = (g\mu^{\epsilon})^{-2/3}. \tag{22b}$$

In Appendix A we show that the diagrammatic contributions are

$$\left.\frac{d\Sigma(\mathbf{q})}{dq_z^2}\right|_{q_z=\mu^2,q_{\perp}=0} = -\frac{g}{16\pi\epsilon}, \tag{23a}$$

$$\left.\frac{d\Sigma(\mathbf{q})}{dq_{\perp}^4}\right|_{q_z=\mu^2,q_{\perp}=0} = (g\mu^{\epsilon})^{-2/3}\frac{g}{32\pi\epsilon}. \tag{23b}$$

Using the conditions on the vertex function we obtain the relations for the renormalization constants in terms of the one-loop diagrammatic corrections. The following relations are correct to lowest order in $\epsilon$:

$$\mathcal{Z} = 1 + \frac{g}{16\pi\epsilon}, \tag{24a}$$

$$\mathcal{Z}_g = 1 + \frac{3g}{64\pi\epsilon}. \tag{24b}$$

### *1. Callan-Symanzik equation*

The renormalized vertex function $\Gamma_r(\mathbf{q})$ satisfies a Callan-Symanzik (CS) equation under a change of length scale $\mu$. We obtain the renormalized elastic moduli from the solution to this equation. The original theory in Eq. (10) did not depend on the length scale $\mu$. We can therefore write the bare vertex function $\Gamma$ in terms of the renormalized vertex function $\Gamma_r$ and find the differential equation obeyed by $\Gamma_r$. Since the variables $u$ and $z$ scale as $u'(\mathbf{x}) = \mathcal{Z}^{1/3}u(\mathbf{x}')$ and $z' = \mathcal{Z}^{1/3}z$, the vertex function must scale as

$$\Gamma(\mathbf{q}_{\perp},q_z,w) = \mathcal{Z}^{-1/3}\Gamma_r(\mathbf{q}_{\perp},\mathcal{Z}^{-1/3}q_z,g,\mu). \tag{25}$$

The CS equation is determined by the condition $\mu d\Gamma/d\mu = 0$. Since the renormalized vertex function can have an explicit as well as an implicit $\mu$ dependence through the functions $\mathcal{Z}$ and $g$, the CS equation for $\Gamma_r$ has three terms

$$\left[\mu\frac{\partial}{\partial\mu} - \frac{\eta(g)}{3}\left(1 + q_z\frac{\partial}{\partial q_z}\right) + \beta(g)\frac{\partial}{\partial g}\right]\Gamma_r = 0, \tag{26}$$

where

$$\beta(g) = \mu\frac{dg}{d\mu}, \tag{27a}$$

$$\eta(g) = \beta(g)\frac{d(\ln\mathcal{Z})}{dg}, \tag{27b}$$

and $q_z\partial/\partial q_z = q'_z\partial/\partial q'_z$ with $q'_z = \mathcal{Z}^{-1/3}q_z$. This equation can be integrated to yield an equation for $\Gamma_r$ as a function of the length scale $\mu$,

$$\Gamma_r(\mathbf{q}_{\perp},q_z,g,\mu) = \exp\left[\frac{1}{3}\int_0^l \eta dl'\right] \times \Gamma_r\left(\mathbf{q}_{\perp},\exp\left[\frac{1}{3}\int_0^l \eta dl'\right]q_z,g(l),\mu_0\right), \tag{28}$$

where $\mu/\mu_0 = e^l$, $\mu d/d\mu = d/dl$, and $g(l)$ must satisfy

$$\frac{dg(l)}{dl} = -\beta(g). \tag{29}$$

At $l=0$ we have set $\Gamma_r(l=0) = \Gamma_r(\mathbf{q}_{\perp},q_z,g_0,\mu_0)$.

Now we must solve for $\beta$ and $\eta$ in terms of $g$ in order to obtain the renormalized vertex function. To find $\beta(g)$, we note that

$$\frac{dw^{3/2}}{dl} = \frac{d}{dl}(g\mu_0^{\epsilon}e^{\epsilon l}\mathcal{Z}_g\mathcal{Z}^{1/2}) = 0. \tag{30}$$

From this relation we find $\beta(g) = -\epsilon/d(\ln Q)/dg$, where $Q = g\mathcal{Z}_g\mathcal{Z}^{1/2}$. We then plug in the relations for $\mathcal{Z}$ and $\mathcal{Z}_g$ and determine $\beta$ and $\eta$ to be

$$\beta(g) = \frac{5}{64\pi}g^2 - \epsilon g, \tag{31a}$$

$$\eta(g) = -\frac{1}{16\pi}g. \tag{31b}$$

In three dimensions $\epsilon=0$. In this case, integration of $dg/dl$ yields

$$g(l) = \frac{g_0}{1 + 5g_0l/64\pi}, \tag{32}$$

where $g_0 \equiv g(0) = w^{3/2}$. The remaining task is simple; we must evaluate the arguments of the exponentials in Eq. (28) to obtain the $l$ dependence of $\Gamma_r$. Since $g \sim 1/l$, the integral of $\eta$ will scale as $\ln l$ and the exponentials of the integral of $\eta$ will give a power-law dependence on $l$. We find that

$$\exp\left[\frac{1}{3}\int_0^l \eta(l')dl'\right]=\left[1+\frac{5g_0}{64\pi}l\right]^{-4/15}\equiv[g/g_0]^{4/15}. \tag{33}$$

### 2. Renormalized elastic constants

The scaling relations in Eqs. (16) and (28) imply that $\Gamma_r$ satisfies

$$\Gamma_r(\mathbf{q}_\perp,q_z,g,\mu)=b^{-4}[g/g_0]^{4/15}\times\Gamma_r(b\mathbf{q}_\perp,b^2[g/g_0]^{4/15}q_z,g,\mu_0 b). \tag{34}$$

We now choose the reference length scale $b=\mu_0^{-1}=(q_z^2+w^{-1}q_\perp^4)^{-1/4}\equiv[h(\mathbf{q})]^{-1}$. This implies that

$$l=\ln\left[\frac{\mu}{h(\mathbf{q})}\right] \tag{35}$$

since $\mu/\mu_0=e^l$. We find the scaling form of the renormalized vertex function

$$\Gamma_r=[h(\mathbf{q})]^4[g/g_0]^{4/15}\Gamma_r\left(\frac{q_\perp}{h(\mathbf{q})},\frac{q_z[g/g_0]^{4/15}}{[h(\mathbf{q})]^2},g,1\right)$$
$$=g^{-2/3}[g/g_0]^{4/15}q_\perp^4+[g/g_0]^{4/5}q_z^2 \tag{36}$$

by squaring the term in the second slot of the renormalized vertex function and adding it to $g^{-2/3}$ times the fourth power of the term in the first slot. We then plug in Eq. (32) for $g$ and transform back to variables with dimension to find the expression for the renormalized vertex function

$$\Gamma_r(\mathbf{q})=B_{\rm sm}\left(1+\frac{5g_0}{64\pi}\ln\left[\frac{\bar{\mu}}{\bar{h}(\mathbf{q})}\right]\right)^{-4/5}q_z^2+K_{\rm sm}\left(1+\frac{5g_0}{64\pi}\ln\left[\frac{\bar{\mu}}{\bar{h}(\mathbf{q})}\right]\right)^{2/5}q_\perp^4, \tag{37}$$

where $g_0=B_{\rm sm}^{1/2}K_{\rm sm}^{-3/2}$, $\bar{\mu}=\mu/B_{\rm sm}^{1/6}$, and $\bar{h}(\mathbf{q})=(q_z^2+\lambda^2q_\perp^4)^{1/4}$ with $\lambda^2=K_{\rm sm}/B_{\rm sm}$. $\bar{\mu}^2$ is a wave number $\Lambda\sim 1/a$ associated with the short distance scale $a$. We identify the renormalized compression and bending moduli $B_{\rm sm}(\mathbf{q})$ and $K_{\rm sm}(\mathbf{q})$ as the coefficients of the $q_z^2$ and $q_\perp^4$ terms, respectively. The renormalized elastic constants scale as powers of logarithms at long wavelengths

$$K_{\rm sm}(\mathbf{q})\sim B_{\rm sm}^{-1/2}(\mathbf{q})\sim\left[\ln\left(\frac{\bar{\mu}}{\bar{h}(\mathbf{q})}\right)\right]^{2/5}, \tag{38}$$

where the long-wavelength regime is defined by wave numbers $q$ that satisfy $\bar{h}(\mathbf{q})\ll\Lambda^{1/2}\exp[-64\pi/5g_0]$. We see that $K_{\rm sm}(\mathbf{q})$ scales to infinity and $B_{\rm sm}(\mathbf{q})$ scales to zero as $q\to 0$.

## III. SLIDING COLUMNAR PHASE WITH RIGID LAYERS

In this section we calculate the logarithmic corrections to the elastic constants for the sliding columnar phase using the dimensional regularization scheme employed in the preceding section. The steps we follow for the dimensional regularization of the SC phase closely resemble those followed for the dimensional regularization of the 3D smectic phase since the two Hamiltonians have similar forms. In this section we assume that each 2D lattice of columns is flat and only allowed to fluctuate in the $z$ direction. We relax this assumption in Sec. IV and find that the renormalized elastic constants are identical to those of the flat theory to lowest order in the coupling between strains in the $y$ and $z$ directions.

### A. Rotationally invariant theory

The rotationally invariant elasticity theory describing the sliding columnar phase was derived previously in [4,5]. We found that a phase with weak positional correlations but strong orientational correlations between neighboring 2D smectic lattices was possible for sufficiently low temperatures. The strong orientational correlations require a rotation modulus in the Landau-Ginzburg-Wilson Hamiltonian that assesses an energy cost for relative rotations of the lattices in addition to the compression and bending energy costs for a single lattice of columns. The Hamiltonian for the idealized sliding columnar phase in three dimensions and in units of $k_BT$ is

$$\mathcal{H}=\frac{1}{2}\int d^3x[Bu_{zz}^2+K(\partial_x^2u_z)^2+K_y(\partial_y\partial_xu_z)^2], \tag{39}$$

where $B$, $K_y$, and $K$ are the compression, rotation, and bending moduli divided by $k_BT$. Symmetry permits additional terms in the Hamiltonian proportional to $K_{zy}(\partial_z\partial_yu_z)^2$ and $K_{zx}(\partial_z\partial_xu_z)^2$. The $K_{zy}$ term measures the energy cost associated with variation in the DNA lattice spacing from layer to layer and the $K_{zx}$ term measures the energy cost associated with the variation in the orientation with strand number of DNA strands within a layer. These terms are, however, subdominant to those kept in Eq. (39) and the couplings $K_{zy}$ and $K_{zx}$ are irrelevant. We will ignore them in what follows. The nonlinear strain $u_{zz}$ is identical to the nonlinear strain for one layer of columns $u_{zz}=\partial_zu_z-(1/2)[(\partial_xu_z)^2+(\partial_zu_z)^2]$. Below we will drop the $(\partial_zu_z)^2$ term from the nonlinear strain since it leads to terms in the nonlinear theory that are also irrelevant with respect to the three harmonic terms in Eq. (39). Therefore, we use the approximate expression

$$u_{zz}\approx\partial_zu_z-\frac{1}{2}(\partial_xu_z)^2. \tag{40}$$

We note that $u_{zz}$ and $\mathcal{H}$ do not possess a shear strain term $(\partial_yu_z)^2$ because neighboring layers of columns can slide relative to one another without energy cost. The absence of the shear energy cost is a unique feature of the sliding columnar elasticity theory. Because the Hamiltonian lacks terms with $y$ derivatives alone, it is invariant with respect to shifts in $u_z$ that are only a function of $y$. Hence $\mathcal{H}[u_z']=\mathcal{H}[u_z]$ with

$$u_z'=u_z+f(y). \tag{41}$$

This invariance restates that there is no energy cost for sliding neighboring layers of columns relative to one another by an arbitrary amount.

### B. Engineering dimensions

We simplify the sliding columnar theory in Eq. (39) by rescaling the lengths so that $B$ and $K_y$ are replaced by unity and the nonlinear form of $u_{zz}$ is preserved. We accomplish this by scaling $u_z$, $y$, and $z$ but not $x$. To implement a dimensional regularization scheme it is necessary to let $x$ become a $(d-2)$-dimensional displacement in the space perpendicular to $y$ and $z$. Rescaled variables are defined via

$$u_z = L_u \tilde{u}_z, \quad x = \tilde{x}, \quad y = L_y \tilde{y}, \quad z = L_z \tilde{z}. \tag{42}$$

We first set $L_u = L_z^{-1}$ to preserve the form of $u_{zz}$ under Eq. (42). We then set the coefficients of $\tilde{u}_{zz}^2$ and $(\partial_{\tilde{y}}\partial_{\tilde{x}}\tilde{u}_z)^2$ to unity by choosing

$$L_y = \left(\frac{K_y^3}{B}\right)^{1/4}, \quad L_z = (K_y B)^{1/4}. \tag{43}$$

The rescaled Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2}\int d^d\tilde{x}[\tilde{u}_{zz}^2 + (\partial_{\tilde{x}}\partial_{\tilde{y}}\tilde{u}_z)^2 + w^{-1}(\partial_{\tilde{x}}^2\tilde{u}_z)^2], \tag{44}$$

with

$$w = \frac{B^{1/2}}{KK_y^{1/2}} \tag{45}$$

and $d = 3-\epsilon$. In the rest of this section we use Eq. (44) and drop the tildes.

We determine the dimension of the scaled variables from the dimensions of the elastic constants in Eq. (39). The dimensions $[B] = L^{-d}$ and $[K_y] = [K] = L^{2-d}$ dictate

$$[u_z] = L^{(3-d)/2}, \quad [x] = L,$$
$$[y] = L^{(d-1)/2}, \quad [z] = L^{(d+1)/2}, \quad [w] = L^{d-3}. \tag{46}$$

Note that $[w]$ scales as $\mu^\epsilon$ with $[\mu] = L^{-1}$ and is relevant below $d=3$.

The engineering dimensions in Eq. (46) imply that the Hamiltonian is invariant under the transformations $\mu \to \mu b$ and

$$u_z(\mathbf{x}) = b^{d_{u_z}} u_z'(\mathbf{x}'), \tag{47}$$

with $x' = b^{-1}x$, $y' = b^{-(d-1)/2}y$, and $z' = b^{-(d+1)/2}z$, i.e., the Hamiltonian obeys

$$\mathcal{H}[u_z, w, \mu] = \mathcal{H}[u_z', wb^\epsilon, \mu b]. \tag{48}$$

This implies that there is a scaling form for the position correlation function $G(\mathbf{x}) = \langle u_z(\mathbf{x}) u_z(0)\rangle$ and the vertex function $\Gamma = G^{-1}$. We find that $\Gamma(\mathbf{q})$ obeys the scaling relation

$$\Gamma(\mathbf{q}, w) = b^{-(d+1)}\Gamma(bq_x, b^{(d-1)/2}q_y, b^{(d+1)/2}q_z, wb^\epsilon). \tag{49}$$

When $d=3$ this reduces to

$$\Gamma(\mathbf{q}, w) = q_x^4 \Gamma(1, q_y/q_x, q_z/q_x^2), \tag{50}$$

which is satisfied by the SC harmonic vertex function $\Gamma = q_z^2 + q_x^2 q_y^2 + w^{-1}q_x^4$.

### C. RG procedure

We now follow closely the RG procedure in Sec. II C. We rescale the lengths and fields, ensure that the SC Hamiltonian has the same form as the unscaled SC Hamiltonian, impose boundary conditions on the vertex function, and determine the renormalization constants in terms of the one-loop diagrammatic corrections. The first step in the process is to rescale lengths such that the renormalized SC Hamiltonian has the same form as Eq. (44). To preserve the form of the nonlinear strain, the $z$ and $u$ rescalings must be inverses of one another and the $y$ rescaling is arbitrary. We therefore introduce two renormalization constants $\mathcal{Z}$ and $\mathcal{Z}_y$ such that

$$u_z(\mathbf{x}) = \mathcal{Z}^{1/3}u_z'(\mathbf{x}') = \mathcal{Z}^{1/3}u_z'(x, \mathcal{Z}_y y, \mathcal{Z}^{1/3}z). \tag{51}$$

This implies that $u_{zz}(\mathbf{x}) = \mathcal{Z}^{2/3}u_{zz}'(\mathbf{x}')$ and $\partial_x\partial_y u_z(\mathbf{x}) = \mathcal{Z}^{1/3}\mathcal{Z}_y \partial_{x'}\partial_{y'}u_z'(\mathbf{x}')$. We also define a unitless coupling constant $g$ and renormalization constant $\mathcal{Z}_g$ by setting

$$w = g\mu^\epsilon \mathcal{Z}^{1/3}\mathcal{Z}_g \mathcal{Z}_y^{-1}. \tag{52}$$

The renormalized Hamiltonian then becomes

$$\mathcal{H}' = \frac{1}{2}\int d^d x'[\mathcal{Z}\mathcal{Z}_y^{-1}(u_{zz}')^2 + \mathcal{Z}^{1/3}\mathcal{Z}_y(\partial_{x'}\partial_{y'}u_z')^2 + (g\mu^\epsilon \mathcal{Z}_g)^{-1}(\partial_{x'}^2 u_z')^2]. \tag{53}$$

We again employ standard RG procedures to calculate $\mathcal{Z}$, $\mathcal{Z}_y$, and $\mathcal{Z}_g$. The renormalization constants are fixed once we impose the following three conditions on the vertex function:

$$\left.\frac{d\Gamma}{dq_z^2}\right|_{q_z=\mu^2, q_{x,y}=0} = 1,$$
$$\left.\frac{d\Gamma}{d(q_x^2 q_y^2)}\right|_{q_z=\mu^2, q_{x,y}=0} = 1, \tag{54}$$
$$\left.\frac{d\Gamma}{dq_x^4}\right|_{q_z=\mu^2, q_{x,y}=0} = (g\mu^\epsilon)^{-1}.$$

(Note that we have dropped the primes on the rescaled Hamiltonian.) The vertex function to one-loop order

$$\Gamma = q_z^2 + q_x^2 q_y^2 + (g\mu^\epsilon)^{-1}q_x^4 + (\mathcal{Z}\mathcal{Z}_y^{-1} - 1)q_z^2 + (\mathcal{Z}^{1/3}\mathcal{Z}_y - 1)q_x^2 q_y^2 + (g\mu^\epsilon)^{-1}(\mathcal{Z}_g^{-1} - 1)q_x^4 + \Sigma(\mathbf{q}) \tag{55}$$

is obtained from Eq. (53) by adding and subtracting $q_z^2 + q_x^2 q_y^2 + (g\mu^\epsilon)^{-1}q_x^4$ and including the one-loop diagrammatic contributions to the vertex function $\Sigma(\mathbf{q})$. In Appendix B we calculate the diagrammatic contributions

$$\left.\frac{d\Sigma}{dq_z^2}\right|_{q_z=\mu^2,q_{x,y}=0}=-\frac{g}{8\pi^2\epsilon}, \tag{56a}$$

$$\left.\frac{d\Sigma}{d(q_x^2q_y^2)}\right|_{q_z=\mu^2,q_{x,y}=0}=\frac{g}{24\pi^2\epsilon}, \tag{56b}$$

$$\left.\frac{d\Sigma}{dq_x^4}\right|_{q_z=\mu^2,q_{x,y}=0}=(g\mu^{\epsilon})^{-1}\frac{g}{12\pi^2\epsilon} \tag{56c}$$

to lowest order in $\epsilon$. From these we determine the renormalization constants to be

$$\mathcal{Z}=1+\frac{g}{16\pi^2\epsilon}, \tag{57a}$$

$$\mathcal{Z}_y=1-\frac{g}{16\pi^2\epsilon}, \tag{57b}$$

$$\mathcal{Z}_g=1+\frac{g}{12\pi^2\epsilon}. \tag{57c}$$

#### 1. Callan-Symanzik equation

The Callan-Symanzik equation is obtained by requiring that the original theory in Eq. (44) be independent of the length scale $\mu$. To ensure this, we set $\mu d\Gamma/d\mu=0$. This can be converted into a differential equation in the renormalized vertex function $\Gamma_r$ using scaling relation

$$\Gamma(\mathbf{q},w)=\mathcal{Z}^{-1/3}\mathcal{Z}_y\Gamma_r(q_x,\mathcal{Z}_y^{-1}q_y,\mathcal{Z}^{-1/3}q_z,g,\mu). \tag{58}$$

From the scaling relation we determine that the CS equation has the four terms

$$\left[\mu\frac{\partial}{\partial\mu}-\frac{\eta(g)}{3}\left(1+q_z\frac{\partial}{\partial q_z}\right)+\eta_y(g)\left(1-q_y\frac{\partial}{\partial q_y}\right)+\beta(g)\frac{\partial}{\partial g}\right]\Gamma_r=0, \tag{59}$$

where $\eta(g)$ and $\beta(g)$ were defined previously in Sec. II C 1 and $\eta_y(g)=\beta(g)d(\ln\mathcal{Z}_y)/dg$. The solution to Eq. (59) is

$$\Gamma_r(\mathbf{q},g,\mu)=\exp\left[\int_0^l\left(\frac{\eta}{3}-\eta_y\right)dl'\right]\Gamma_r\left(q_x,\exp\left[\int_0^l\eta_y dl'\right]q_y,\exp\left[\frac{1}{3}\int_0^l\eta dl'\right]q_z,g,\mu_0\right), \tag{60}$$

with $\Gamma_r(l=0)=\Gamma_r(\mathbf{q},g_0,\mu_0)$ and $\mu/\mu_0=e^l$.

The coupling constant $w$ must be independent of the length scale $l$. This condition yields a differential equation for the dimensionless constant $g$ whose solution is

$$g(l)=\frac{g_0}{1+g_0l/6\pi^2}. \tag{61}$$

This equation in turn determines the $l$ dependence of $\eta$ and $\eta_y$ since they are both proportional to $g$. We find

$$\eta(g)=-\eta_y(g)=\frac{g}{16\pi^2} \tag{62}$$

and thus these scale as $1/l$ at long wavelengths.

#### 2. Renormalized elastic constants

Using Eq. (61) for $g(l)$ and the relations for $\eta(g)$ and $\eta_y(g)$ in Eq. (62), we obtain the scaling form of the renormalized vertex function

$$\Gamma_r(\mathbf{q},g,\mu)=b^{-4}[g/g_0]^{1/2}\Gamma_r(bq_x,bq_y[g/g_0]^{-3/8},b^2q_z[g/g_0]^{1/8},g,\mu_0 b). \tag{63}$$

To set the length scale, we choose

$$b=\mu_0^{-1}=(q_z^2+q_x^2q_y^2+w^{-1}q_x^4)^{-1/4}\equiv[h(\mathbf{q})]^{-1}. \tag{64}$$

It follows that

$$l=\ln\left[\frac{\mu}{h(\mathbf{q})}\right] \tag{65}$$

since $\mu$ and $l$ are related via $\mu/\mu_0=e^l$. We then substitute Eq. (61) for $g$ and transform back to variables with dimension to obtain the following expression for $\Gamma_r(\mathbf{q})$:

$$\Gamma_r(\mathbf{q})=B\left(1+\frac{g_0}{6\pi^2}\ln\left[\frac{\overline{\mu}}{\overline{h}(\mathbf{q})}\right]\right)^{-3/4}q_z^2+K_y\left(1+\frac{g_0}{6\pi^2}\ln\left[\frac{\overline{\mu}}{\overline{h}(\mathbf{q})}\right]\right)^{1/4}q_x^2q_y^2+K\left(1+\frac{g_0}{6\pi^2}\ln\left[\frac{\overline{\mu}}{\overline{h}(\mathbf{q})}\right]\right)^{1/2}q_x^4, \tag{66}$$

where $g_0=B^{1/2}/KK_y^{1/2}$, $\overline{\mu}=\mu/(K_yB)^{1/8}$, and $\overline{h}(\mathbf{q})=(q_z^2$

$+\lambda_y^2 q_x^2 q_y^2 + \lambda^2 q_x^4)^{1/4}$, with $\lambda_y^2 = K_y/B$ and $\lambda^2 = K/B$. $\overline{\mu}^2$ is an upper momentum cutoff $\Lambda \sim 1/a$ associated with the short-distance scale $a$. We can now identify the $q$-dependent elastic constants and determine their scaling as $q$ tends to zero. At long wavelengths such that $\overline{h}(\mathbf{q}) \ll \Lambda^{1/2}\exp[-6\pi^2/g_0]$ the ln term dominates and we find

$$K_y(\mathbf{q}) \sim K^{1/2}(\mathbf{q}) \sim B^{-1/3}(\mathbf{q}) \sim \left[\ln\left(\frac{\overline{\mu}}{\overline{h}(\mathbf{q})}\right)\right]^{1/4}. \tag{67}$$

We see that $B(\mathbf{q})$ scales to zero and $K(\mathbf{q})$ and $K_y(\mathbf{q})$ scale to infinity as $q \to 0$. Also note in Table I that the exponents of the logarithmic power laws of $B(\mathbf{q})$ and $K(\mathbf{q})$ are different from those of $B_{\rm sm}(\mathbf{q})$ and $K_{\rm sm}(\mathbf{q})$, but the signs of the respective exponents are the same.

## IV. SLIDING COLUMNAR PHASE WITH FLUCTUATING LIPID BILAYERS

In the preceding section we considered a model for lamellar DNA-lipid complexes in which lipid bilayers were treated as rigid planes and no displacements of DNA lattices in the $y$ direction were allowed. In physically realized complexes, lipid bilayers can undergo shape fluctuations and DNA lattices can undergo $y$ displacements. We can parameterize the shape of the $n$th bilayer by a height function $h_n(x,z)$, which in the continuum limit becomes $h(\mathbf{x}) = h_{y/a}(x,z)$. The $y$ displacement of the DNA lattices in the continuum limit is $u_y(\mathbf{x})$. At long wavelengths the displacements $h(\mathbf{x})$ and $u_y(\mathbf{x})$ are locked together. The lock-in occurs because there is an energy cost for translating each lattice of columns and the lipid bilayers by different constant amounts in the $y$ direction. (See Fig. 1.) We can therefore describe long-wavelength elastic distortions and fluctuations of the sliding columnar phase in terms of a Landau-Ginzburg-Wilson elastic Hamiltonian expressed in terms of displacements $u_z$ and $u_y$:

$$\begin{aligned}\mathcal{H}_b[u_y,u_z] = \frac{1}{2}\int d^3x [&B^z u_{zz}^2 + K_{xx}^z(\partial_x^2 u_z)^2 \\ &+ K_{xy}^z(\partial_x\partial_y u_z)^2 + B^y u_{yy}^2 + K_{xx}^y(\partial_x^2 u_y)^2 \\ &+ K_{xz}^y(\partial_x\partial_z u_y)^2 + K_{zz}^y(\partial_z^2 u_y)^2 + 2B^{yz}u_{yy}u_{zz}],\end{aligned} \tag{68}$$

where $u_{yy}$ and $u_{zz}$ are nonlinear strains. We define $\mathcal{H}_b$ to have units of $k_BT$ and therefore the constants appearing in this equation are the compression and bending moduli divided by $k_BT$. The first three terms in Eq. (68) were discussed previously in Sec. III as the $u_z$ theory for the sliding columnar phase without fluctuations of the lipid bilayers. The next four terms are the compression and bending energies for an anisotropic 3D smectic with layers parallel to the $xz$ plane. The bending energy is anisotropic due to the presence of the DNA columns. The final term is a coupling of the nonlinear strains $u_{yy}$ and $u_{zz}$.

The form of the nonlinear strains depends on whether Eulerian or Lagrangian coordinates are used [14]. We find it convenient to use a mixed parameterization in which $x$ and $z$ are Eulerian coordinates specifying a position in space and $y = na$ is a Lagrangian coordinate specifying the layer number. In Appendix D we derive the nonlinear strains $u_{zz}$ and $u_{yy}$ for this mixed parametrization. To quadratic order in gradients of $u_y$ and $u_z$, we find

$$u_{yy} = \partial_y u_y - \frac{1}{2}[(\partial_x u_y)^2 + (\partial_z u_y)^2 - (\partial_y u_y)^2], \tag{69a}$$

$$u_{zz} = \partial_z u_z - \frac{1}{2}[(\partial_x u_z)^2 + (\partial_z u_z)^2 - (\partial_z u_y)^2]. \tag{69b}$$

Note that the nonlinear strain $u_{zz}$ does not contain the shear strain term proportional to $(\partial_y u_z)^2$. Thus, layer fluctuations do not modify the essential invariance $u_z' \to u_z + f(y)$ of the sliding columnar phase to the order considered here [15]. In what follows, we will truncate the nonlinear strains to

$$u_{yy} \approx \partial_y u_y, \tag{70a}$$

$$u_{zz} \approx \partial_z u_z - \frac{1}{2}(\partial_x u_z)^2 \tag{70b}$$

since the other nonlinear terms are irrelevant with respect to the sliding columnar harmonic terms in Eq. (68).

The goal of this section is to calculate the Grinstein-Pelcovits renormalization of the eight elastic constants found in the theory of the sliding columnar phase with lipid bilayer fluctuations. Since the nonlinear strains do not introduce a $(\partial_y u_z)^2$ term, we do not expect the bilayer fluctuations to alter the renormalization of the SC elastic constants in the simplified theory of Sec. III to lowest order in $B^{yz}$. We will again use dimensional regularization to calculate the renormalization. The format will closely parallel the previous SC calculation. We first determine which of the harmonic terms in Eq. (68) are relevant and drop irrelevant terms. We then rescale lengths and fields, ensure that the Hamiltonian retains its unscaled form, impose boundary conditions on the vertex function, and calculate the renormalization constants. The renormalization constants then determine the scaling form of the vertex function.

### A. Engineering dimensions

We begin by rescaling the lengths and the fields in $\mathcal{H}_b$. In addition to the rescalings in Sec. III B, we also rescale $u_y$ according to

$$u_y = L_{u_y}\tilde{u}_y. \tag{71}$$

We first impose the conditions of Sec. III B, i.e., we set the coefficients of $\tilde{u}_{zz}^2$ and $(\partial_{\tilde{x}}\partial_{\tilde{y}}u_z)^2$ to unity and ensure that both terms in the nonlinear strain $u_{zz}$ scale the same way. As an added constraint, we set the coefficient of $\tilde{u}_{yy}^2$ to unity. These conditions fix

$$L_{u_y}=\left(\frac{K_{xy}^z}{B^z}\right)^{1/4}\frac{1}{(B^y)^{1/2}},$$

$$L_y=\left(\frac{(K_{xy}^z)^3}{B^z}\right)^{1/4}, \tag{72}$$

$$L_z=L_{u_z}^{-1}=(K_{xy}^z B^z)^{1/4}.$$

Once we plug in these scaling lengths, the rescaled Hamiltonian becomes

$$\begin{aligned}\mathcal{H}_b=\frac{1}{2}\int d^d\tilde{x}[\tilde{u}_{zz}^2+(\partial_{\tilde{x}}\partial_{\tilde{y}}u_z)^2+w^{-1}(\partial_{\tilde{x}}^2\tilde{u}_z)^2\\ +(\partial_{\tilde{y}}\tilde{u}_y)^2+2v(\partial_{\tilde{y}}\tilde{u}_y)\tilde{u}_{zz}+v_1(\partial_{\tilde{x}}^2\tilde{u}_y)^2\\ +v_2(\partial_{\tilde{x}}\partial_{\tilde{z}}\tilde{u}_y)^2+v_3(\partial_{\tilde{z}}^2\tilde{u}_y)^2],\end{aligned} \tag{73}$$

with

$$w=\frac{(B^z)^{1/2}}{K_{xx}^z(K_{xy}^z)^{1/2}},\quad v=\frac{B^{yz}}{(B^yB^z)^{1/2}},$$

$$v_1=\frac{K_{xx}^y(K_{xy}^z)^{3/2}}{B^y(B^z)^{1/2}},\quad v_2=\frac{K_{xz}^yK_{xy}^z}{B^yB^z}, \tag{74}$$

$$v_3=\frac{K_{zz}^y(K_{xy}^z)^{1/2}}{(B^z)^{3/2}B^y}.$$

(It is again necessary to let $x$ represent a $d-2$ displacement with $d=3-\epsilon$.) The dimensions of the scaled variables and the $w$ and $v$ coefficients are determined using Eq. (72) and the dimensions of the compression and bending moduli $[B]=L^{-d}$ and $[K]=L^{2-d}$. (Note we have dropped the tildes on the scaled variables in the following discussion.) We find

$$\begin{aligned}&[u_y]=L^{(1-d)/2},\quad [v]=L^0,\quad [v_1]=L^{5-d},\\ &[v_2]=L^4,\quad [v_3]=L^{d+3},\end{aligned} \tag{75}$$

while the dimensions of $u_z$, $y$, $z$, and $w$ were given previously in Eq. (46). Note that $v$ does not scale with length. Also note that the coefficients $v_1$, $v_2$, and $v_3$ are irrelevant when $d=3$. We drop the irrelevant terms and arrive at the simplified Hamiltonian

$$\begin{aligned}\mathcal{H}_b=\frac{1}{2}\int d^dx[u_{zz}^2+(\partial_x\partial_yu_z)^2+w^{-1}(\partial_x^2u_z)^2\\ +(\partial_yu_y)^2+2v(\partial_yu_y)u_{zz}].\end{aligned} \tag{76}$$

### B. RG procedure

The present RG procedure will be similar to those employed in Secs. II C and III C, except we now have two coupling constants $w$ and $v$ instead of one. We will show that the inclusion of $v$ does not alter the renormalization of the sliding columnar elastic constants to lowest order in $v$. As before, we rescale the fields and lengths and seek a renormalized Hamiltonian with the same form as Eq. (76). We scale $y$, $z$, and $u_z$ as we did previously in Eq. (51) and $u_y$ by $\tilde{\mathcal{Z}}^{1/2}$ as

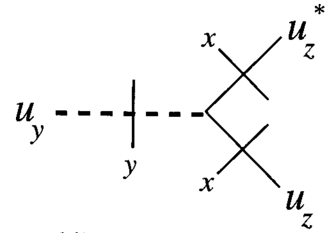


FIG. 2. Schematic diagram of the additional relevant nonlinear term $\partial_yu_y(\partial_xu_z)^2$ generated by the sliding columnar theory with lipid bilayer fluctuations. The symbols $x$ and $y$ written adjacent to the dividing lines represent $x$ and $y$ derivatives of the respective fields. The $u_y$ field is denoted by a dashed line, while $u_z$ is denoted by a solid line.

$$u_y(\mathbf{x})=\tilde{\mathcal{Z}}^{1/2}u_y'(\mathbf{x}')=\tilde{\mathcal{Z}}^{1/2}u_y'(x,\mathcal{Z}_yy,\mathcal{Z}^{1/3}z). \tag{77}$$

The rescaled Hamiltonian $\mathcal{H}_b'$ looks similar to Eq. (53) with two additional terms due to fluctuations of the bilayers. We drop the primes on the variables and find

$$\begin{aligned}\mathcal{H}_b=\frac{1}{2}\int d^dx[\mathcal{Z}\mathcal{Z}_y^{-1}u_{zz}^2+\mathcal{Z}^{1/3}\mathcal{Z}_y(\partial_x\partial_yu_z)^2\\ +(g\mu^\epsilon\mathcal{Z}_g)^{-1}(\partial_x^2u_z)^2+\mathcal{Z}^{-1/3}\mathcal{Z}_y\tilde{\mathcal{Z}}(\partial_yu_y)^2\\ +2\overline{v}\mathcal{Z}_v(\partial_yu_y)u_{zz}],\end{aligned} \tag{78}$$

where

$$\overline{v}\mathcal{Z}_v=v\tilde{\mathcal{Z}}^{1/2}\mathcal{Z}^{1/3} \tag{79}$$

and $\mathcal{Z}_g$ was defined previously.

Boundary conditions imposed on the vertex functions $\Gamma_{ij}(\mathbf{q})$ with $i,j=y,z$ ensure that the Hamiltonian retains its original form in Eq. (76) after rescaling. The vertex function is defined by $\Gamma_{ij}(\mathbf{q})=G_{ij}^{-1}(\mathbf{q})$ with $G_{ij}(\mathbf{x})=\langle u_i(\mathbf{x})u_j(0)\rangle$. The conditions imposed on $\Gamma_{zz}$ are identical to those given in Eq. (54); these are augmented by two conditions on $\Gamma_{yz}$ and $\Gamma_{yy}$,

$$\begin{aligned}\left.\frac{d\Gamma_{yz}}{d(q_yq_z)}\right|_{q_z=\mu^2,q_{x,y}=0}=2\overline{v},\\ \left.\frac{d\Gamma_{yy}}{dq_y^2}\right|_{q_z=\mu^2,q_{x,y}=0}=1.\end{aligned} \tag{80}$$

Once we impose these conditions on the vertex functions, we solve for the $\mathcal{Z}$'s in terms of the one-loop diagrammatic contributions $\Sigma_{ij}$, where, for instance, $\Sigma_{zz}$ is the one-loop correction to the vertex function $\Gamma_{zz}$. The diagrammatic corrections arise from the quadratic term in $u_{zz}$. $u_{zz}^2$ generates $\partial_zu_z(\partial_xu_z)^2$, which was already present in the theory with $u_y=0$. The coupling of $u_{yy}$ to $u_{zz}$ generates a new nonlinear term $\partial_yu_y(\partial_xu_z)^2$. This term is shown schematically in Fig. 2. There are six more one-loop diagrams in addition to the

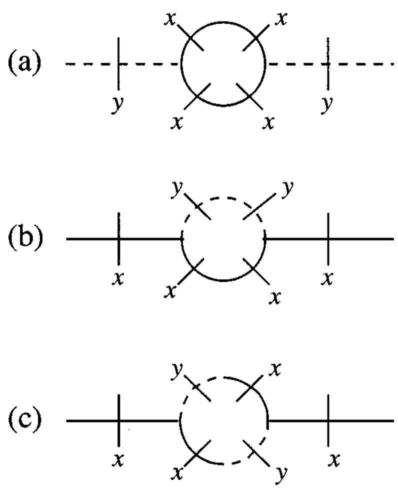


FIG. 3. Three diagrams that can be formed by contracting $\partial_y u_y(\partial_x u_z)^2$ with itself. The only diagram that contributes to the renormalization of $B^y$ is pictured in (a). The diagrams in (b) and (c) contribute to the renormalization of both $K^z_{xx}$ and $K^z_{xy}$.

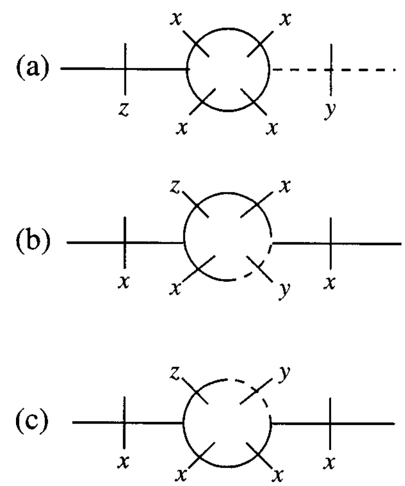


FIG. 4. Three diagrams that can be formed by contracting $\partial_z u_z(\partial_x u_z)^2$ with $\partial_y u_y(\partial_x u_z)^2$. The only diagram that contributes to the renormalization of $B^{yz}$ is pictured in (a). The diagrams pictured in (b) and (c) contribute to the renormalization of both $K^z_{xx}$ and $K^z_{xy}$.

three diagrams of the rigid sliding columnar theory; these are shown in Figs. 3 and 4. The diagrams in Fig. 3 arise from contractions of $\partial_y u_y(\partial_x u_z)^2$ with itself and the diagrams in Fig. 4 arise from contractions of $\partial_y u_y(\partial_x u_z)^2$ with $\partial_z u_z(\partial_x u_z)^2$.

The one-loop diagrammatic corrections $\Sigma_{zz}$ are easy to calculate since the form of the propagator $G_{zz}$ is unchanged from its form in the rigid sliding columnar theory. The form is not changed, but the compression modulus $B$ is renormalized by a factor of $1-\overline{v^2}$. The one-loop diagrammatic corrections to $\Gamma_{zz}$ are shown to lowest order in $\epsilon$:

$$\left.\frac{d\Sigma_{zz}}{dq_z^2}\right|_{q_z=\mu^2,q_{x,y}=0} = -\frac{g}{8\pi^2\epsilon}\frac{1}{\sqrt{1-\overline{v^2}}},$$

$$\left.\frac{d\Sigma_{zz}}{d(q_x^2 q_y^2)}\right|_{q_z=\mu^2,q_{x,y}=0} = \frac{g}{24\pi^2\epsilon}\sqrt{1-\overline{v^2}}, \qquad (81)$$

$$\left.\frac{d\Sigma_{zz}}{dq_x^4}\right|_{q_z=\mu^2,q_{x,y}=0} = (g\mu^\epsilon)^{-1}\frac{g}{12\pi^2\epsilon}\sqrt{1-\overline{v^2}}.$$

These expressions reduce to those found for the rigid theory when $\overline{v}=0$.

The calculation of one-loop diagrammatic corrections to $\Gamma_{yz}$ and $\Gamma_{yy}$ is similarly straightforward. $\Sigma_{yz}$ is given by the diagram in Fig. 4(a). This amplitude is proportional to $\overline{v}$ since it is formed by contracting $\partial_y u_y(\partial_x u_z)^2$ with $\partial_z u_z(\partial_x u_z)^2$. $\Sigma_{yy}$ is given by the diagram in Fig. 3(a); it must be proportional to $\overline{v^2}$ since it is formed by contracting $\partial_y u_y(\partial_x u_z)^2$ with itself. The one-loop corrections to $\Gamma_{yz}$ and $\Gamma_{yy}$ are given below to lowest order in $\epsilon$:

$$\left.\frac{d\Sigma_{yz}}{d(q_y q_z)}\right|_{q_z=\mu^2,q_{x,y}=0} = -\frac{g\overline{v}}{8\pi^2\epsilon}\frac{1}{\sqrt{1-\overline{v^2}}},$$

$$\left.\frac{d\Sigma_{yy}}{dq_y^2}\right|_{q_z=\mu^2,q_{x,y}=0} = -\frac{g\overline{v^2}}{8\pi^2\epsilon}\frac{1}{\sqrt{1-\overline{v^2}}}. \qquad (82)$$

We then use the conditions imposed on the vertex functions in Eqs. (54) and (80) and the one-loop diagrammatic corrections in Eqs. (81) and (82) to find the renormalization constants (the $\mathcal{Z}$'s) in terms of $g$ and $\overline{v}$. We find that the relations for $\mathcal{Z}$, $\mathcal{Z}_y$, and $\mathcal{Z}_g$ are unchanged to zeroth order in $\overline{v}$. $\tilde{\mathcal{Z}}$ and $\mathcal{Z}_v$ also have terms that are independent of $\overline{v}$ as shown below to lowest order in $\epsilon$:

$$\tilde{\mathcal{Z}} \approx 1+\frac{g}{12\pi^2\epsilon}, \quad \mathcal{Z}_v \approx 1+\frac{g}{8\pi^2\epsilon}. \qquad (83)$$

The variation of $g$ and $\overline{v}$ with the length scale $\mu$ is obtained by enforcing that both bare coupling constants do not depend on $\mu$, i.e., we set $\mu dw/d\mu = \mu dv/d\mu = 0$. These two requirements determine the recursion relations for $g$ and $\overline{v}$; we find that $dg/dl$ is unchanged to lowest order in $\overline{v}$ and

$$\frac{d\overline{v}}{dl} = -\frac{g\overline{v}}{16\pi^2}. \qquad (84)$$

The zeroth-order solution for $g$ was found previously in Eq. (61); we plug this solution into Eq. (84) and find

$$\overline{v}(l) = \frac{\overline{v}_0}{[1+g_0 l/6\pi^2]^{3/8}}, \qquad (85)$$

where $\overline{v}_0 = B^{yz}/\sqrt{B^y B^z}$ and $g_0 = \sqrt{B^z/K^z_{xy}}/K^z_{xx}$.

### C. Renormalized elastic constants

We found in Secs. II C 2 and III C 2 that the renormalized elastic constants are obtained by solving the Callan-Symanzik equation for the renormalized vertex function. We find the CS equations for $\Gamma^r_{ij}$ using the following scaling equations that relate the bare and renormalized vertex functions:

$$\Gamma_{zz}(\mathbf{q},w,v)=\mathcal{Z}^{-1/3}\mathcal{Z}_y\Gamma^r_{zz}(\mathbf{q}',g,\overline{v},\mu), \tag{86a}$$

$$\Gamma_{yy}(\mathbf{q},w,v)=\tilde{\mathcal{Z}}^{-1}\mathcal{Z}_y^{-1}\mathcal{Z}^{1/3}\Gamma^r_{yy}(\mathbf{q}',g,\overline{v},\mu), \tag{86b}$$

$$\Gamma_{yz}(\mathbf{q},w,v)=\tilde{\mathcal{Z}}^{-1/2}\mathcal{Z}_y\Gamma^r_{yz}(\mathbf{q}',g,\overline{v},\mu). \tag{86c}$$

Equation (86a) yields a CS equation identical to Eq. (59) to lowest order in $\overline{v}$ and thus the renormalized elastic constants $B^z(\mathbf{q})$, $K^z_{xx}(\mathbf{q})$, and $K^z_{xy}(\mathbf{q})$ are identical to those obtained in Eq. (66) using the $u_y=0$ theory. The fact that the elastic constants are identical to zeroth order in $\overline{v}$ is a consequence of the fact that the nonlinear term proportional to $\overline{v}$ does not introduce any harmonic terms that were not already present in the theory without $u_y$ fluctuations. We also find that the coefficient of $\Gamma^r_{yy}(\mathbf{q}')$ is unity to lowest order $\overline{v}$ and hence the vertex function $\Gamma_{yy}$ does not rescale. As a result, $B^y=B^y(l=0)$ plus higher-order terms in $\overline{v}$.

We do, however, find a nontrivial renormalization of $B^{yz}$. The scaling relation in Eq. (86c) leads to a CS equation for $\Gamma^r_{yz}$ with a similar form to the one found in Eq. (59). We find

$$\left[\mu\frac{\partial}{\partial\mu}-\frac{\tilde{\eta}(g)}{2}-\frac{\eta(g)}{3}\left(q_z\frac{\partial}{\partial q_z}\right)+\eta_y(g)\left(1-q_y\frac{\partial}{\partial q_y}\right)+\beta(g)\frac{\partial}{\partial g}\right]\Gamma_r=0 \tag{87}$$

to zeroth order in $\overline{v}$, where

$$\tilde{\eta}(g)=\beta(g)\frac{d(\ln\tilde{\mathcal{Z}})}{dg}=\frac{g}{12\pi^2} \tag{88}$$

and $\eta$ and $\eta_y$ were defined previously. The solution to Eq. (87) can be transcribed from Eq. (60) and is

$$\Gamma^r_{yz}(\mathbf{q},g,\overline{v}(g),\mu)=\exp\left[\int_0^l\left(\frac{\tilde{\eta}}{2}-\eta_y\right)dl'\right]\Gamma^r_{yz}\left(q_x,\exp\left[\int_0^l\eta_y dl'\right]q_y,\exp\left[\frac{1}{3}\int_0^l\eta dl'\right]q_z,g,\mu_0\right). \tag{89}$$

Since $\eta$, $\eta_y$, and $\tilde{\eta}$ scale as $1/l$, the integrals in the arguments of the exponentials scale logarithmically with $l$. Thus the exponentials yield power laws in $g$ and we find, for example,

$$\exp\left[\int_0^l\left(\frac{\tilde{\eta}}{2}-\eta_y\right)dl'\right]=\left[\frac{g(l)}{g_0}\right]^{5/8}. \tag{90}$$

The renormalized vertex function in Eq. (89) obeys a scaling form analogous to the one obeyed by the renormalized sliding columnar vertex function in Eq. (63). We find

$$\Gamma^r_{yz}(\mathbf{q},g,\mu)=b^{-3}[g/g_0]^{5/8}\Gamma^r_{yz}(bq_x,bq_y[g/g_0]^{-3/8},b^2q_z[g/g_0]^{1/8},g,\mu_0 b), \tag{91}$$

where the $b^{-3}$ prefactor is present because $y$ scales as $b$ and $z$ scales as $b^2$. We then choose $b=\mu_0^{-1}=[q_z^2+q_x^2q_y^2+w^{-1}q_x^4]^{-1/4}\equiv[h(\mathbf{q})]^{-1}$ to match the conventions of Sec. III C 2, substitute Eq. (61) for $g/g_0$, and return to variables with dimension. The renormalized vertex function becomes

$$\Gamma^r_{yz}(\mathbf{q})=2B^{yz}\left[1+\frac{g_0}{6\pi^2}\ln\left(\frac{\overline{\mu}}{\overline{h}(\mathbf{q})}\right)\right]^{-3/4}q_yq_z, \tag{92}$$

where $\overline{\mu}$ and $\overline{h}(\mathbf{q})$ were defined previously. The renormalized elastic constant $B^{yz}(\mathbf{q})$ is the coefficient of $q_yq_z$ in the above expression. Therefore, we find that both $B^z$ and $B^{yz}$ scale to zero logarithmically with $\mathbf{q}$ at long wavelengths defined by $\overline{h}(\mathbf{q})\ll\Lambda^{1/2}\exp[-6\pi^2/g_0]$.

## V. CONCLUSION

We have calculated the Grinstein-Pelcovits renormalization of the elastic constants for the sliding columnar phase. We first used a simplified model of the sliding columnar phase in which the DNA columns were prevented from fluctuating perpendicular to the lipid layers. We found that the elastic constants scaled as powers of $\ln[1/q]$ at long wavelengths. In particular, we found that the compression modulus $B$ scales to zero and the rotation and bending moduli $K_y$ and $K$ scale to infinity as $q$ tends to zero. We then added in perpendicular fluctuations of the columns perturbatively and found that the above results were unchanged to lowest order in the coupling between strains parallel and perpendicular to the lipid layers. We employed dimensional regularization in our RG analysis of the sliding columnar phase to ensure rotational invariance. RG schemes that break rotational invariance, such as the momentum-shell technique, did not yield correct results.

## ACKNOWLEDGMENTS

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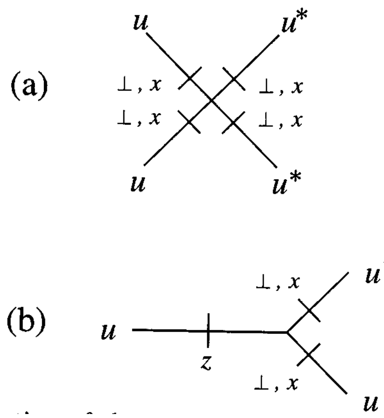


FIG. 5. Schematic representation of the two relevant nonlinear terms in both the 3D smectic and sliding columnar elasticity theories. The perpendicular derivatives ($\perp$) correspond to the 3D smectic theory and the $x$ derivatives to the sliding columnar theory. The term $(\partial_{\perp,x}u)^4$ is pictured in (a) and the term $(\partial_z u)(\partial_{\perp,x}u)^2$ is pictured in (b). The symbols $\perp$, $x$, and $z$ represent $\perp$, $x$, and $z$ derivatives of the $u$ field. The diagram with four $u$ fields in (a) does not contribute to the renormalization to one-loop order; only contractions of (b) with itself contribute.

## APPENDIX A: EVALUATION OF THE 3D SMECTIC ONE-LOOP DIAGRAMS

Our task in this appendix is to calculate $\Sigma(\mathbf{q})$ defined in Sec. II C as the one-loop diagrammatic corrections to $\Gamma(\mathbf{q})$, the vertex function for the 3D smectic. These corrections arise from the nonlinear terms in the Hamiltonian in Eq. (10). The two nonlinear terms are $\partial_z u_z(\nabla_\perp u)^2/2$ and $(\nabla_\perp u)^4/8$ (shown schematically in Fig. 5) and only contractions of the former contribute to the renormalization to one-loop order. The three possible contractions are shown in Fig. 6. The diagrammatic corrections $\Sigma(\mathbf{q})$ can be expressed as

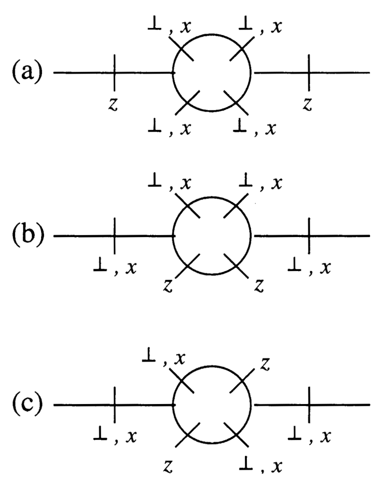


FIG. 6. Three one-loop diagrams that contribute to the renormalization of the 3D smectic and sliding columnar elastic constants. These diagrams are formed by contracting $\partial_z u(\partial_{\perp,x}u)^2$ with itself. The diagram in (a) contributes terms proportional to $q_z^2$ since a factor of $q_z$ is on each external leg. The diagrams in (b) and (c) contribute terms proportional to $q_\perp^4$ in the 3D smectic theory and terms proportional to $q_x^2 q_y^2$ and $q_x^4$ in the sliding columnar theory since these diagrams have $q_\perp^2$ or $q_x^2$ on the external legs.

$$\Sigma(\mathbf{q})=\Pi_1(\mathbf{q})q_z^2+\Pi_2(\mathbf{q})q_\perp^4\equiv\Sigma_1(\mathbf{q})+\Sigma_2(\mathbf{q}). \quad \text{(A1)}$$

Note that we have separated the $q_z^2$ and $q_\perp^4$ dependence of $\Sigma(\mathbf{q})$ so that to lowest order in $\mathbf{q}$

$$\left.\frac{d\Sigma}{dq_z^2}\right|_{q_z=\mu^2,q_\perp=0}=\left.\frac{d\Sigma_1}{dq_z^2}\right|_{q_z=\mu^2,q_\perp=0} \quad \text{(A2)}$$

and

$$\left.\frac{d\Sigma}{dq_\perp^4}\right|_{q_z=\mu^2,q_\perp=0}=\left.\frac{d\Sigma_2}{dq_\perp^4}\right|_{q_z=\mu^2,q_\perp=0}. \quad \text{(A3)}$$

The contributions of $d\Sigma_2/dq_z^2$ to $d\Sigma/dq_z^2$ and of $d\Sigma_1/dq_\perp^4$ to $d\Sigma/dq_\perp^4$ at the special point $q_z=\mu^2$ and $q_\perp=0$ are higher order in $\epsilon$ than the contributions in Eqs. (A2) and (A3). We begin by calculating $\Sigma_1(\mathbf{q})$.

### 1. Calculation of $\Sigma_1(\mathbf{q})$

The diagram in Fig. 6(a) alone contributes to $\Sigma_1(\mathbf{q})$ since it is the only one with $q_z^2$ on the external legs. To evaluate the integrals in the perturbation theory, we use dimensional regularization, i.e., we take $d=3-\epsilon$, set the cutoff to infinity, and look for the $1/\epsilon$ terms. $\Sigma_1(\mathbf{q})$ is obtained by calculating the $q_z^2$ contribution from the integral

$$\Sigma_1(\mathbf{q})=-\frac{q_z^2}{2}\int_{-\infty}^{\infty}\frac{d^{3-\epsilon}k}{(2\pi)^{3-\epsilon}}[(q_\perp+k_\perp)_i(q_\perp+k_\perp)_j \times k_{\perp i}k_{\perp j}G(\mathbf{k}+\mathbf{q})G(-\mathbf{k})], \quad \text{(A4)}$$

where $i,j=x,y$ and

$$G(\mathbf{q})=\frac{1}{q_z^2+w^{-1}q_\perp^4}. \quad \text{(A5)}$$

The coefficient of the $q_z^2$ term in Eq. (A4) is $\Pi_1(\mathbf{q})$. We can then approximate $\Sigma_1(\mathbf{q})$ by writing $\Sigma_1(\mathbf{q})=q_z^2\Pi_1(q_\perp=0,q_z)$ plus higher-order terms in $q_\perp$ that vanish when we apply the boundary condition in Eq. (22a). We obtain $\Pi_1(q_z)$ by setting $q_\perp=0$ in the integral on the right-hand side of Eq. (A4).

To evaluate the integral, we first combine the denominators of $G(\mathbf{k}+\mathbf{q})$ and $G(-\mathbf{k})$ employing the identity

$$\frac{1}{(k_z+q_z)^2+w^{-1}k_\perp^4}\frac{1}{[k_z^2+w^{-1}k_\perp^4]}=\int_0^1 dx\frac{1}{[(k_z+xq_z)^2+x(1-x)q_z^2+w^{-1}k_\perp^4]^2}. \quad \text{(A6)}$$

We then change variables to $k_z'=k_z+xq_z$ and perform the integration over $k_z'$. We find that $\Sigma_1(\mathbf{q})$ can be written in terms of the integral $J(4,3,x,q_z)$ with $J(s,v,x,q_z)$ defined by

$$J(s,v,x,q_z)=\int_0^\infty dk_\perp k_\perp^{1-\epsilon}\frac{k_\perp^s}{[x(1-x)q_z^2+w^{-1}k_\perp^4]^{v/2}}$$
$$=\frac{w^{v/2}}{4\Gamma(v/2)}\Gamma\left(\frac{1}{4}(2v-s-2+\epsilon)\right)$$
$$\times\Gamma\left(\frac{1}{4}(s+2-\epsilon)\right)$$
$$\times[x(1-x)wq_z^2]^{(s-2v+2-\epsilon)/4}, \quad \text{(A7)}$$

where $\Gamma(x)$ is the gamma function evaluated at $x$. The expression for $\Sigma_1(\mathbf{q})$ is simple when expressed in terms of the integral $J(4,3,x,q_z)$; we find

$$\Sigma_1(\mathbf{q})=-\frac{q_z^2}{16\pi}\int_0^1 dx\, J(4,3,x,q_z). \quad \text{(A8)}$$

From Eq. (A7) we know that the most dominant term in $J(4,3,x,q_z)$ scales as $1/\epsilon$ and thus

$$\Sigma_1(\mathbf{q})=-\frac{w^{3/2}}{16\pi\epsilon}q_z^2(wq_z^2)^{-\epsilon/4} \quad \text{(A9)}$$

plus higher-order terms in $\epsilon$. We can also write $\Sigma_1(\mathbf{q})$ as

$$\left.\frac{d\Sigma_1(\mathbf{q})}{dq_z^2}\right|_{q_z=\mu^2,q_\perp=0}=-\frac{g}{16\pi\epsilon} \quad \text{(A10)}$$

when we replace $w$ by $(g\mu^\epsilon)^{2/3}$.

#### 2. Calculation of $\Sigma_2(\mathbf{q})$

$\Sigma_2(\mathbf{q})$ is determined by calculating the $q_\perp^4$ contributions from the diagrams in Figs. 6(b) and 6(c). $\Sigma_2(\mathbf{q})$ is the $q_\perp^4$ part of the integral

$$\Sigma_2(\mathbf{q})=-q_{\perp i}q_{\perp j}\int\frac{d^{3-\epsilon}k}{(2\pi)^{3-\epsilon}}[(k_z+q_z)^2k_{\perp i}k_{\perp j}$$
$$+(k_z+q_z)(k_\perp+q_\perp)_jk_{\perp i}k_z]$$
$$\times G(\mathbf{k}+\mathbf{q})G(-\mathbf{k}). \quad \text{(A11)}$$

The $q_\perp^4$ contributions come from expanding $G(\mathbf{k}+\mathbf{q})$ to second order in $q_\perp$; we see from Eq. (A11) that we need both the first- and second-order terms in the expansion. The coefficient of the $q_\perp^4$ term in the above expansion is $\Pi_2(q_\perp=0,q_z)$ and thus $\Sigma_2(\mathbf{q})=q_\perp^4\Pi_2(q_z)$ plus higher-order terms in $q_\perp$ that vanish when we apply the boundary condition in Eq. (22b).

The first and second terms in the integrand of Eq. (A11) correspond to the diagrams in Figs. 6(b) and 6(c), respectively. We break up the integral so that $\Sigma_2(\mathbf{q})=\Sigma_2^b(\mathbf{q})+\Sigma_2^c(\mathbf{q})$ and we first calculate $\Sigma_2^b(\mathbf{q})$.

$$\Sigma_2^b(\mathbf{q})=-\frac{1}{2}q_{\perp i}q_{\perp j}q_{\perp_l}q_{\perp_m}$$
$$\times\int_{-\infty}^\infty\frac{dk_z}{2\pi}\int\frac{d\Omega}{(2\pi)^{2-\epsilon}}dk_\perp k_\perp^{1-\epsilon}$$
$$\times\left[(k_z+q_z)^2k_{\perp i}k_{\perp j}G(-\mathbf{k})\left.\frac{d^2G(\mathbf{k}+\mathbf{q})}{dq_{\perp_l}dq_{\perp_m}}\right|_{q_\perp=0}\right], \quad \text{(A12)}$$

where $\Omega$ is the solid angle in $2-\epsilon$ dimensions and the second derivative of $G$ gives the coefficient of the quadratic term in the expansion of $G(\mathbf{k}+\mathbf{q})$. We then remove the angular dependence by integrating over $\Omega$ and using the two identities

$$\int\frac{d\Omega}{(2\pi)^{2-\epsilon}}k_{\perp i}k_{\perp j}=\frac{S_{2-\epsilon}}{2-\epsilon}k_\perp^2\delta_{ij} \quad \text{(A13)}$$

and

$$\int\frac{d\Omega}{(2\pi)^{2-\epsilon}}k_{\perp i}k_{\perp j}k_{\perp l}k_{\perp m}$$
$$=\frac{S_{2-\epsilon}}{(2-\epsilon)^3}k_\perp^4(\delta_{ij}\delta_{lm}+\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}), \quad \text{(A14)}$$

where $\delta_{ij}$ is the Kronecker delta and $S_d=\Omega/(2\pi)^d=2\pi^{d/2}/(2\pi)^d\Gamma(d/2)$ with $d=2-\epsilon$. We are interested in the lowest-order terms in $\epsilon$ and hence will use $S_{2-\epsilon}\approx(2\pi)^{-1}$ below. We then change variables to $k_z'=k_z+q_z$ and combine the denominators of $G(-\mathbf{k})$ and $G(\mathbf{k}+\mathbf{q})$ using an identity similar to Eq. (A6):

$$\frac{1}{(k_z-q_z)^2+w^{-1}k_\perp^4}\ \frac{1}{[k_z^2+w^{-1}k_\perp^4]^n}=\Gamma(n+1)\int_0^1 dx\frac{f_n(x)}{[(k_z-xq_z)^2+x(1-x)q_z^2+w^{-1}k_\perp^4]^{n+1}}, \quad \text{(A15)}$$

where $n=2,3$ and

$$f_n(x)=\begin{cases}1-x, & n=2\\ (1-x)^2/2, & n=3.\end{cases} \tag{A16}$$

We change variables again to $k_z''=k_z+xq_z$ and integrate over $k_z''$; we find that $\Sigma_2^b(\mathbf{q})$ can be written in terms of the integrals $J(s,v,x,q_z)$ defined previously in Eq. (A7):

$$\Sigma_2^b(\mathbf{q})=-\frac{w^{-1}}{32\pi}q_\perp^4\int_0^1 dx[-5(1-x)J(4,3,x,q_z)$$
$$-15x^2(1-x)q_z^2J(4,5,x,q_z)$$
$$+9w^{-1}(1-x)^2J(8,5,x,q_z)$$
$$+45w^{-1}x^2(1-x)^2q_z^2J(8,7,x,q_z)]. \tag{A17}$$

$J(4,3,x,q_z)$ and $J(8,5,x,q_z)$ have terms proportional to $1/\epsilon$, but $J(4,5,x,q_z)$ and $J(8,7,x,q_z)$ do not. We keep the terms that are proportional to $1/\epsilon$ and drop the others. In the last step we perform the $x$ integration and find

$$\Sigma_2^b(\mathbf{q})=-\frac{w^{1/2}}{64\pi\epsilon}q_\perp^4(wq_z^2)^{-\epsilon/4} \tag{A18}$$

plus higher-order terms in $\epsilon$.

We next obtain $\Sigma_2^c(\mathbf{q})$ by calculating the $q_\perp^4$ contributions from the diagram in Fig. 6(c). $\Sigma_2^c(\mathbf{q})$ can be written in terms of the integral

$$\Sigma_2^c(\mathbf{q})=-q_{\perp i}q_{\perp j}\int\frac{d\Omega}{(2\pi)^{2-\epsilon}}dk_\perp k_\perp^{1-\epsilon}$$
$$\times\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\Bigg[k_z(k_z+q_z)G(-\mathbf{k})$$
$$\times\Bigg(k_{\perp i}q_{\perp j}q_{\perp_l}\frac{dG(\mathbf{k}+\mathbf{q})}{dq_{\perp_l}}\Bigg|_{q_\perp=0}$$
$$+k_{\perp i}k_{\perp j}\frac{q_{\perp_l}q_{\perp_m}}{2}\frac{d^2G(\mathbf{k}+\mathbf{q})}{dq_{\perp_l}dq_{\perp_m}}\Bigg|_{q_\perp=0}\Bigg)\Bigg]. \tag{A19}$$

The first and second derivatives of $G$ give the coefficients of the linear and quadratic terms in $q_\perp$ in the expansion of $G(\mathbf{k}+\mathbf{q})$. We then follow a procedure similar to the one employed to find $\Sigma_2^b(\mathbf{q})$, i.e., we change variables to $k_z'=k_z+q_z$, combine the denominators of $G(\mathbf{k}+\mathbf{q})$ and $G(-\mathbf{k})$, and integrate over $\Omega$. The remaining integrals in Eq. (A19) are over $k_\perp$ and $x$. We then integrate over $k_\perp$ and write $\Sigma_2^c(\mathbf{q})$ in terms of $J(s,v,x,q_z)$; we find

$$\Sigma_2^c(\mathbf{q})=-\frac{w^{-1}}{32\pi}q_\perp^4\int_0^1 dx[-9(1-x)J(4,3,x,q_z)$$
$$+27x(1-x)^2q_z^2J(4,5,x,q_z)$$
$$+9w^{-1}(1-x)^2J(8,5,x,q_z)$$
$$-45w^{-1}x(1-x)^2q_z^2J(8,7,x,q_z)]. \tag{A20}$$

Only $J(4,3,x,q_z)$ and $J(8,5,x,q_z)$ have terms proportional to $1/\epsilon$. We keep these terms and perform the integration over $x$ to find

$$\Sigma_2^c(\mathbf{q})=\frac{3w^{1/2}}{64\pi\epsilon}q_\perp^4(wq_z^2)^{-\epsilon/4}. \tag{A21}$$

We obtain $\Sigma_2(\mathbf{q})$ by adding $\Sigma_2^b(\mathbf{q})$ and $\Sigma_2^c(\mathbf{q})$ in Eqs. (A18) and (A21) to yield

$$\frac{d\Sigma_2(\mathbf{q})}{dq_\perp^4}\Bigg|_{q_z=\mu^2,q_\perp=0}=(g\mu^\epsilon)^{-2/3}\frac{g}{32\pi\epsilon} \tag{A22}$$

once we set $w=(g\mu^\epsilon)^{2/3}$ and ignore higher-order terms in $\epsilon$.

## APPENDIX B: EVALUATION OF THE SLIDING COLUMNAR LOOP DIAGRAMS

The aim of this appendix is to calculate $\Sigma(\mathbf{q})$, the one-loop diagrammatic corrections to the vertex function for the sliding columnar phase. The rotationally invariant theory given in Eq. (44) contains two relevant nonlinear terms $\partial_z u_z(\partial_x u_z)^2$ and $(\partial_x u_z)^4$. These terms are pictured schematically in Fig. 5. From this figure we see that only contractions of $\partial_z u_z(\partial_x u_z)^2$ renormalize the elastic constants to one-loop order. The three possible contractions are shown in Fig. 6. $\Sigma(\mathbf{q})$ has $q_z^2$, $q_x^2q_y^2$, and $q_x^4$ contributions and we will calculate each separately below. To do this, we express $\Sigma(\mathbf{q})$ as

$$\Sigma(\mathbf{q})=\Pi_1(\mathbf{q})q_z^2+\Pi_2(\mathbf{q})q_x^2q_y^2+\Pi_3(\mathbf{q})q_x^4$$
$$\equiv\Sigma_1(\mathbf{q})+\Sigma_2(\mathbf{q})+\Sigma_3(\mathbf{q}). \tag{B1}$$

We have separated the $q_z^2$, $q_x^2q_y^2$, and $q_x^4$ dependences so that, for instance,

$$\frac{d\Sigma}{dq_x^4}\Bigg|_{q_z=\mu^2,q_\perp=0}=\frac{d\Sigma_3}{dq_x^4}\Bigg|_{q_z=\mu^2,q_\perp=0}. \tag{B2}$$

As in Appendix A, we use dimensional regularization to calculate the integrals.

### 1. Calculation of $\Sigma_1(\mathbf{q})$

The $q_z^2$ contribution to $\Sigma(\mathbf{q})$ results from squaring the diagram pictured in Fig. 5(b) and contracting both pairs of $x$ derivatives. This leaves $q_z$ on each external leg as shown in Fig. 6(a). $\Sigma_1(\mathbf{q})$ is the $q_z^2$ part of the integral

$$\Sigma_1(\mathbf{q}) = -\frac{q_z^2}{2}\int \frac{d^{3-\epsilon}k}{(2\pi)^{3-\epsilon}} \times[(q_x+k_x)^2 k_x^2 G(\mathbf{q}+\mathbf{k})G(-\mathbf{k})], \tag{B3}$$

where

$$G(\mathbf{q}) = \frac{1}{q_z^2+q_x^2q_y^2+w^{-1}q_x^4}. \tag{B4}$$

The coefficient of the $q_z^2$ in the above integral is $\Pi_1(\mathbf{q})$ and thus $\Sigma_1(\mathbf{q}) = q_z^2\Pi_1(q_{x,y}=0,q_z)$ plus higher-order terms in $q_x$ and $q_y$ that vanish when we apply the boundary condition in Eq. (54). Thus $\Sigma_1(\mathbf{q})$ is obtained by setting $q_x=q_y=0$ in Eq. (B3). We find

$$\Sigma_1(\mathbf{q}) = -\frac{w^{-1/2}}{2}\frac{q_z^2}{(2\pi)^{3-\epsilon}}\int dk_x dk_z d^{1-\epsilon}k_y \times\left[\frac{k_x^4}{k_z^2+w^{-1}k_x^2k_\perp^2}\times\frac{1}{(q_z+k_z)^2+w^{-1}k_x^2k_\perp^2}\right], \tag{B5}$$

where we have changed variables to $k_y = w^{-1/2}k_y'$ and dropped the prime. The first step in evaluating this integral is to combine the two denominators in Eq. (B6) using the identity in Eq. (A6) with $k_\perp^4$ replaced by $k_x^2k_\perp^2$. We then perform the integration over $k_z$ and find that $\Sigma_1(\mathbf{q})$ can be written in terms of the integral $I(4,0,3,x,q_z)$, where

$$\begin{aligned} I(s,t,v,x,q_z) &= \int_0^\infty dk_x dk_y \frac{k_x^s k_y^{t-\epsilon}}{[x(1-x)q_z^2+w^{-1}k_x^2k_\perp^2]^{v/2}} \\ &= \frac{w^{v/2}}{8\Gamma(v/2)}\Gamma\left(\frac{1}{2}(t+1-\epsilon)\right)\Gamma\left(\frac{1}{4}(s-t+\epsilon)\right) \\ &\quad\times\Gamma\left(\frac{1}{4}(2v-t-s-2+\epsilon)\right) \\ &\quad\times[x(1-x)wq_z^2]^{(s+t-2v+2-\epsilon)/4}. \end{aligned} \tag{B6}$$

We give the most general form for the integrals over $k_x$ and $k_y$ since we will need these integrals later when we calculate $\Sigma_2(\mathbf{q})$ and $\Sigma_3(\mathbf{q})$. We find

$$\Sigma_1(\mathbf{q}) = \frac{-w^{-1/2}}{8\pi^2}q_z^2\int_0^1 dx\, I(4,0,3,x,q_z) \tag{B7}$$

and

$$\Sigma_1(\mathbf{q}) = -\frac{w}{8\pi^2\epsilon}q_z^2(wq_z^2)^{-\epsilon/4} \tag{B8}$$

since $I(4,0,3,x,q_z)\propto 1/\epsilon$. We then set $w=g\mu^\epsilon$ to find $\Sigma_1(\mathbf{q})$ as a function of $g$,

$$\left.\frac{d\Sigma_1(\mathbf{q})}{dq_z^2}\right|_{q_z=\mu^2,q_{x,y}=0} = -\frac{g}{8\pi^2\epsilon}. \tag{B9}$$

### 2. Calculation of $\Sigma_2(\mathbf{q})$

Both the $q_x^2q_y^2$ and $q_x^4$ contributions to $\Sigma(\mathbf{q})$ come from the diagrams with $x$ derivatives on the external legs. The two contributing diagrams are shown in Figs. 6(b) and 6(c). Their sum $\mathcal{S}$ is given by

$$\mathcal{S} = -q_x^2\int\frac{d^{3-\epsilon}k}{(2\pi)^3}[(k_z+q_z)^2k_x^2 +(q_z+k_z)(q_x+k_x)k_zk_x]G(\mathbf{k}+\mathbf{q})G(-\mathbf{k}). \tag{B10}$$

We find the $q_x^2q_y^2$ terms by expanding $G(\mathbf{k}+\mathbf{q})$ to second order in $q_y$. We see that only the quadratic term in the expansion contributes. Higher-order terms will vanish when we apply the second boundary condition in Eq. (54). We then follow a procedure similar to the one employed to calculate the $q_\perp^4$ contribution to the 3D smectic vertex function in Appendix A. We find that $\Sigma_2(\mathbf{q})$ can be written in terms of the integrals $I(s,t,v,x,q_z)$ as

$$\begin{aligned} \Sigma_2(\mathbf{q}) = -\frac{w^{-1/2}}{8\pi^2}q_x^2q_y^2\int_0^1 dx[&-2(1-x)I(4,0,3,x,q_z) \\ &+6w^{-1}(1-x)^2I(6,2,5,x,q_z) \\ &-3xq_z^2(2x-1)(1-x)I(4,0,5,x,q_z) \\ &+15w^{-1}xq_z^2(2x-1)(1-x)^2I(6,2,7,x,q_z)]. \end{aligned} \tag{B11}$$

We look for the leading-order terms in $\epsilon$ in Eq. (B11); $I(4,0,3,x,q_z)$ and $I(6,2,5,x,q_z)$ have leading-order terms proportional to $1/\epsilon$, while $I(4,0,5,x,q_z)$ and $I(6,2,7,x,q_z)$ do not and are dropped. After integrating Eq. (B11) over $x$ we obtain

$$\Sigma_2(\mathbf{q}) = \frac{w}{24\pi^2\epsilon}q_x^2q_y^2(wq_z^2)^{-\epsilon/4} \tag{B12}$$

and

$$\left.\frac{d\Sigma_2}{d(q_x^2q_y^2)}\right|_{q_z=\mu^2,q_{x,y}=0} = \frac{g}{24\pi^2\epsilon}. \tag{B13}$$

### 3. Calculation of $\Sigma_3(\mathbf{q})$

$\Sigma_3(\mathbf{q})$ is obtained by calculating the terms proportional to $q_x^4$ in Eq. (B10). We obtain these terms by expanding $G(\mathbf{k}+\mathbf{q})$ to second order in $q_x$ and noting that both first- and second-order terms in the expansion contribute. Note that higher-order terms in the expansion will vanish once we apply the third boundary condition in Eq. (54). We calculate the $q_x^4$ contributions from Figs. 6(b) and 6(c) separately and define $\Sigma_3(\mathbf{q})\equiv\Sigma_3^b(\mathbf{q})+\Sigma_3^c(\mathbf{q})$. We first calculate the contribution from Fig. 6(b). Using the same procedure as the one employed to calculate the $q_x^2q_y^2$ contribution to $\Sigma(\mathbf{q})$, we find that $\Sigma_3^b(\mathbf{q})$ can be written in terms of the integral $I(s,t,v,x,q_z)$,

$$\Sigma_3^b(\mathbf{q}) = -\frac{w^{-3/2}}{8\pi^2} q_x^4 \int_0^1 dx\{-(1-x)\times[6I(4,0,3,x,q_z)+I(2,2,3,x,q_z)+18x^2q_z^2 I(4,0,5,x,q_z)+3x^2q_z^2 I(2,2,5,x,q_z)]+3w^{-1}(1-x)^2[4I(8,0,5,x,q_z)+20x^2q_z^2 I(8,0,7,x,q_z)+4I(6,2,5,x,q_z)+20x^2q_z^2 I(6,2,7,x,q_z)+I(4,4,5,x,q_z)+5x^2q_z^2 I(4,4,7,x,q_z)]\}. \quad \text{(B14)}$$

We note that three of the integrals in Eq. (B14), $I(4,0,5,x,q_z)$, $I(8,0,7,x,q_z)$, and $I(6,2,7,x,q_z)$, have leading-order terms that scale as $\epsilon^0$ and are dropped. Two integrals $I(2,2,3,x,q_z)$ and $I(4,4,5,x,q_z)$ have $1/\epsilon^2$ as well as $1/\epsilon$ terms, while the remaining five integrals $I(4,0,3,x,q_z)$, $I(2,2,5,x,q_z)$, $I(8,0,5,x,q_z)$, $I(6,2,5,x,q_z)$, and $I(4,4,7,x,q_z)$ have leading-order contributions that scale as $1/\epsilon$. We collect terms and perform the $x$ integration to find

$$\Sigma_3^b(\mathbf{q}) = -\frac{1}{8\pi^2\epsilon} q_x^4 (wq_z^2)^{-\epsilon/4}\left[\frac{1}{\epsilon}+\ln[2]-\frac{1}{12}\right]. \quad \text{(B15)}$$

Note that the dominant contribution to $\Sigma_3^b$ is of order $\epsilon^{-2}$ rather than $\epsilon^{-1}$. The undesirable $\epsilon^{-2}$ term and the $\ln[2]/\epsilon$ term will be cancelled by terms in $\Sigma_3^c$. The term proportional to $\ln[2]/\epsilon$ originates from the integrals $I(2,2,3,x,q_z)$ and $I(4,4,5,x,q_z)$. This can be seen by expanding $I(4,4,5,x,q_z)$ in powers of $\epsilon$; we find

$$I(4,4,5,x,q_z) = \frac{2w^{-5/2}}{\epsilon^2}\left(1-\frac{\epsilon}{2}\frac{\Gamma'(5/2)}{\Gamma(5/2)}+\frac{\epsilon}{2}\frac{\Gamma'(1)}{\Gamma(1)}\right)\times[x(1-x)wq_z^2]^{-\epsilon/4} \quad \text{(B16)}$$

to $O(1/\epsilon)$, where $\Gamma'(x)$ is the derivative of the $\Gamma$ function evaluated at $x$. The logarithm arises from evaluating the derivative of the $\Gamma$ function at a half integer. For example, $\Gamma'(5/2)/\Gamma(5/2) = -\gamma + 8/3 - 2\ln[2]$ where $\gamma$ is the Euler-Mascheroni constant.

We can also write $\Sigma_3^c(\mathbf{q})$ in terms of the integrals $I(s,t,w,x,q_z)$. We obtain

$$\Sigma_3^c(\mathbf{q}) = -\frac{w^{-3/2}}{8\pi^2} q_x^4 \int_0^1 dx\{(1-x)[-10I(4,0,3,x,q_z)+30x(1-x)q_z^2 I(4,0,5,x,q_z)-3I(2,2,3,x,q_z)+9x(1-x)q_z^2 I(2,2,5,x,q_z)]+3w^{-1}(1-x)^2[4I(8,0,5,x,q_z)-20x(1-x)q_z^2 I(8,0,7,x,q_z)+4I(6,2,5,x,q_z)-20x(1-x)q_z^2 I(6,2,7,x,q_z)+I(4,4,5,x,q_z)-5x(1-x)q_z^2 I(4,4,7,x,q_z)]\}, \quad \text{(B17)}$$

which becomes

$$\Sigma_3^c(\mathbf{q}) = -\frac{1}{8\pi^2\epsilon} q_x^4 (wq_z^2)^{-\epsilon/4}\left[-\frac{1}{\epsilon}-\ln[2]-\frac{7}{12}\right] \quad \text{(B18)}$$

when only terms proportional to $1/\epsilon^2$ and $1/\epsilon$ are retained. We see that when we add Eq. (B15) to Eq. (B18), the terms proportional to $1/\epsilon^2$ and $\ln[2]/\epsilon$ cancel and we are left with

$$\Sigma_3(\mathbf{q}) = \frac{1}{12\pi^2\epsilon} q_x^4 (wq_z^2)^{-\epsilon/4} \quad \text{(B19)}$$

and

$$\left.\frac{d\Sigma_3(\mathbf{q})}{dq_x^4}\right|_{q_z=\mu^2, q_{x,y}=0} = (g\mu^\epsilon)^{-1}\frac{g}{12\pi^2\epsilon}. \quad \text{(B20)}$$

## APPENDIX C: FINITE WAVE-NUMBER CUTOFF

In this appendix we show that employing a finite cutoff leads to ambiguities when we evaluate the sliding columnar one-loop diagrams. These diagrams are shown in Fig. 6; Fig. 6(a) contributes to $\Sigma_1(\mathbf{q})$ and both Figs. 6(b) and 6(c) contribute to $\Sigma_2(\mathbf{q})$ and $\Sigma_3(\mathbf{q})$. The ambiguous result is that we obtain different answers for $\Sigma(\mathbf{q})$ depending on whether external momentum $q$ is sent through the top or bottom part of the internal loop. The ambiguity develops when momentum $q_x$ appears in the internal loop and the top and bottom paths through the internal loop are different. The diagram that causes this ambiguity is the $q_x^4$ part of Fig. 6(b). We can see this by calculating the $q_x^4$ corrections to the vertex function, $\Sigma_3^b(\text{top})$ and $\Sigma_3^b(\text{bot})$, which result from sending $\mathbf{k}+\mathbf{q}$ through the top (bottom) sections of the internal loop:

$$\Sigma_3^b(\text{top}) = -q_x^2 \int_\Lambda \frac{d^3k}{(2\pi)^3}\times[k_z^2(k_x+q_x)^2 G(-\mathbf{k})G(\mathbf{k}+\mathbf{q})] \quad \text{(C1)}$$

and

$$\Sigma_3^b(\text{bot}) = -q_x^2 \int_\Lambda \frac{d^3k}{(2\pi)^3}\times[k_x^2(k_z+q_z)^2 G(-\mathbf{k})G(\mathbf{k}+\mathbf{q})], \quad \text{(C2)}$$

where $\Lambda$ is a finite-wave-number cutoff and $G(\mathbf{q})$ was defined previously in Eq. (B4). With $\Lambda \neq \infty$,

$$\Sigma_3^b(\text{top}) \neq \Sigma_3^b(\text{bot}). \quad \text{(C3)}$$

If we employ dimensional regularization instead and send $\Lambda \to \infty$, these top and bottom amplitudes are identical.

## APPENDIX D: DERIVATION OF THE NONLINEAR STRAINS IN THE PRESENCE OF FLEXIBLE MEMBRANES

In this appendix we derive expressions for the nonlinear strains $u_{yy}(\mathbf{x})$ and $u_{zz}(\mathbf{x})$ introduced in Eqs. (69a) and (69b) for the case of flexible membranes. A complete description of lamellar DNA-lipid complexes requires separate coordinates for each membrane and each DNA molecule. Displacements of membranes and DNA molecules parallel to the membrane normals (along the $y$ direction when the membranes are flat) are locked together. We can therefore model the complexes as a stack of membranes each with a one-dimensional mass-density wave representing the DNA lattice just above it. We employ mixed Lagrangian-Eulerian variables in which the coordinate $y = na$ specifying the layer or membrane number is a Lagrangian variable and the coordinates $(x,z) \equiv \mathbf{r}$ are Eulerian variables specifying positions in a fixed projection plane. The positions of mass points on membrane $n$ are then given by

$$\mathbf{R}_n(\mathbf{r}) = x\hat{x} + z\hat{z} + [na + u_y(na,\mathbf{r})]\hat{y}. \tag{D1}$$

The density in membrane $n$ can be expanded as $\rho_n(\mathbf{r}) = \rho_n^0 + \psi_n(\mathbf{r}) + \psi_n^*(\mathbf{r})$, where $\rho_n^0$ is a constant, $\psi_n(\mathbf{r}) = |\psi_n| e^{i\phi_n(\mathbf{r})}$, and

$$\phi_n(\mathbf{r}) = k_0[z - u_z(na,\mathbf{r})] \tag{D2}$$

with $k_0 = 2\pi/d$.

To construct the strain variable $u_{yy}(\mathbf{x})$ with $\mathbf{x} = (y,\mathbf{r})$, we introduce the distance $l_n(\mathbf{r},\mathbf{r}')$ between points $\mathbf{r}$ on membrane $n$ and $\mathbf{r}'$ on membrane $n+1$ via

$$l_n^2(\mathbf{r},\mathbf{r}') = |\mathbf{R}_{n+1}(\mathbf{r}') - \mathbf{R}_n(\mathbf{r})|^2. \tag{D3}$$

The shortest distance between a point $\mathbf{r}$ on membrane $n$ and any point on membrane $n+1$ is then

$$l_n^2(\mathbf{r}) = \min_{\mathbf{r}'} l_n^2(\mathbf{r},\mathbf{r}'). \tag{D4}$$

The strain variable $u_{yy}$ is defined as

$$u_{yy}(\mathbf{x}) = \lim_{a\to 0} \frac{1}{2a^2}[l_{y/a}^2(\mathbf{r}) - a^2]. \tag{D5}$$

This quantity is by construction invariant with respect to global rotations of the entire system. To evaluate $u_{yy}(\mathbf{x})$, we expand $\mathbf{R}_{n+1}(\mathbf{r}') - \mathbf{R}_n(\mathbf{r})$ to lowest order in $\delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$ and $a$:

$$\mathbf{R}_{n+1}(\mathbf{r}') = \mathbf{R}_n(\mathbf{r}) + a[1 + \partial_y u_y(\mathbf{x})]\hat{y} + \delta r^\mu \mathbf{e}_\mu, \tag{D6}$$

where $\mu = x,z$, $\mathbf{e}_\mu = \partial_\mu \mathbf{R}_n(\mathbf{x})$ is a covariant tangent-plane vector of the $n$th surface, and $u_y(\mathbf{x}) = u_y(na,\mathbf{r})$. Then

$$l_n^2(\mathbf{r},\mathbf{r}') = a^2(1+\partial_y u_y)^2 + 2a(1+\partial_y u_y)\,\delta r^\mu \partial_\mu u_y + g_{\mu\nu}\delta r^\mu \delta r^\nu, \tag{D7}$$

where $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$ is the metric tensor of the $n$th surface and we used $\hat{y}\cdot\mathbf{e}_\mu = \partial_\mu u_y$. We then minimize $l_n^2(\mathbf{r},\mathbf{r}')$ over $\delta r^\mu$ and obtain

$$\delta r^\mu = -a(1+\partial_y u_y) g^{\mu\nu}\partial_\nu u_y \tag{D8}$$

and

$$l_{y/a}^2(\mathbf{r}) = a^2(1+\partial_y u_y)^2(1 - g^{\mu\nu}\partial_\mu u_y \partial_\nu u_y). \tag{D9}$$

Finally, using $g^{\mu\nu} = (g_{\mu\nu})^{-1}$ where

$$g_{\mu\nu} = \delta_{\mu\nu} + \partial_\mu u_y \partial_\nu u_y, \tag{D10}$$

we obtain

$$u_{yy}(\mathbf{x}) = \frac{1}{2}\left[\frac{(1+\partial_y u_y)^2}{1+(\nabla u_y)^2} - 1\right] \approx \partial_y u_y - \frac{1}{2}[(\partial_x u_y)^2 + (\partial_z u_y)^2 - (\partial_y u_y)^2], \tag{D11}$$

with $\nabla = (\partial_x, 0, \partial_z)$. It is straightforward to verify that $u_{yy}(\mathbf{x}) = 0$ for a uniform rotation of the entire system. For example, a rotation of the system about the $z$ axis by $\theta$ produces strains $\partial_y u_y = 1/\cos\theta - 1$ and $\partial_x u_y = \tan\theta$, which cause $u_{yy}$ to vanish.

The strain $u_{zz}(\mathbf{x})$ can also be defined in a rotationally invariant way via

$$u_{zz}(\mathbf{x}) = \frac{1}{2k_0^2}[k_0^2 - g^{\mu\nu}\partial_\mu \phi(\mathbf{x})\partial_\nu \phi(\mathbf{x})], \tag{D12}$$

where $\phi(\mathbf{x}) = \phi_{y/a}(\mathbf{r})$ is defined in Eq. (D2). To quadratic order in $\partial_\mu u_z$ and $\partial_\mu u_y$, the nonlinear strain $u_{zz}$ is

$$u_{zz}(\mathbf{x}) \approx \partial_z u_z - \frac{1}{2}[(\partial_x u_z)^2 + (\partial_z u_z)^2 - (\partial_z u_y)^2], \tag{D13}$$

where $u_z(\mathbf{x}) = u_z(y,\mathbf{r})$.


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